AD-A127 639 OPTIMUM MULTISENSOR MULTITARGET TIME DELAY ESTIMATION 1/3
(U) NAVAL UNDERWATER SYSTEMS CENTER NEW LONDON CT NEW
LONDON LAB L C NG ET AL. 20 APR 83 NUSC-TR-6757

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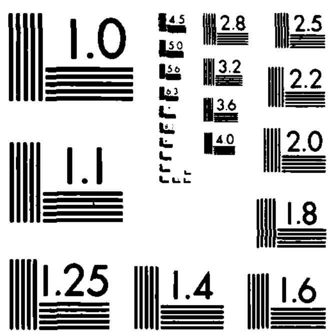

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NUSC Technical Report 6757
20 April 1983

Optimum Multisensor, Multitarget Time Delay Estimation

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Preface

The preparation of this report was supported in part by the following NUSC programs: (1) Project No. A75210, "Time Delay Estimation and Tracking," Principal Investigator, Lawrence C. Ng (Code 3213); an (2) Project No. A12212, "WAA Localization," Principal Investigator, P. Dean (Code 3212). The sponsoring activity was NAVSEA PMS-409, Program Principal, John Neeley (NAVSEA PMS-409). In addition, this study was also supported by ONR under Contract N00014-78-C-0529 (Program Principal, James Smith).

The authors appreciate the technical inputs provided by Profs. C. Knapp and C. Nikias from the University of Connecticut; Drs. N. Owsley, J. Ianniello, and J. Hassab, Messrs. R. Garber, S. Kessler, J. Dlubac, and G. Drury from the Naval Underwater Systems Center; and Mr. H. Jarvis and Dr. C. Wenk from Analysis & Technology, Inc.

The Technical Reviewer for this report was Dr. G. C. Carter, Code 3331.

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REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER TR 6757	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) OPTIMUM MULTISENSOR, MULTITARGET TIME DELAY ESTIMATION		5. TYPE OF REPORT & PERIOD COVERED
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Lawrence C. Ng, Y. Bar-Shalom		8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS Naval Underwater Systems Center, New London Laboratory, New London, Connecticut 06320		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS A75210 A12212
11. CONTROLLING OFFICE NAME AND ADDRESS Naval Sea Systems Command (PMS-409) Washington, D.C. 20362		12. REPORT DATE 20 April 1983
		13. NUMBER OF PAGES 209
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION / DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Optimum Multisensor
Time Delay Estimation

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This study presents the results of an investigation of the optimum signal processor for passive time delay estimation in a multisensor, multitarget environment. The target-sensor geometry is assumed stationary over the observation time interval. The performance of the optimum processor in terms of the Cramer-Rao Lower Bound is presented. The relation between time delay estimation and localization parameter estimation is explored. A comparison between optimum and suboptimum implementations is considered. The derivation of the optimum processor for joint time delay and power spectral estimation is

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also presented. Methods for improved time delay estimation in the presence of interference are investigated. Finally, a simple analytical expression for the Cramer-Rao Lower Bound for the joint cross-sensor time delay estimate from M sensor arrays is derived.

SUMMARY

Under the assumption of a Stationary Parameter Long Observation Time (SPLOT) process, the optimum multisensor, multitarget time delay processor was derived from a Maximum Likelihood (ML) viewpoint. The resulting processor was obtained by reducing the vector likelihood equation via straightforward, but somewhat tedious, manipulations to the simplest form. The optimum multisensor multitarget processor provided the basis for the subsequent detailed studies on time delay vector estimation, localization parameter estimation, power spectral estimation, suboptimum processor realization and post Generalized Cross Correlation (GCC) multitarget parameter estimation. Major findings of this study are summarized below.

1. The optimum multisensor, multitarget time delay estimator is a highly coupled, multichannel signal processor. The estimator's performance in terms of the Cramer-Rao Lower Bound (CRLB) was evaluated for the following cases: one target and two sensors, two targets and two sensors, and one target and M sensors. For the one target, two sensor case, the optimum multisensor multitarget processor reduces to the GCC studied by Knapp and Carter (Reference 1). For the two sensor, two target case, the resulting processor was studied in detail. A closed form analytical expression was obtained. The results of the study showed that the optimum processor provides a significant improvement over a conventional GCC processor. For the one target, M sensor case, the matrix CRLB was obtained.

   The results of this study indicated that for an M sensor array, the M-1 inter-sensor time delays can be obtained from M-1 correlations. This is a significant reduction from Hahn's approach (Reference 25), where a total of M (M-1)/2 correlations are required. Additionally, the results showed that the variance of the time delay estimate between any two sensors decreases with M, the total number of sensors.

2. The localization parameter estimation was examined in detail for a three-sensor array from two perspectives: (a) a one-step focused beamformer approach where a direct range and bearing estimate is sought, and (b) a two-step time delay to range and bearing approach where time delay estimates are first sought and range and bearing estimates are obtained via a geometric mapping. The study indicated that both approaches yield an identical performance bound. However, for practical considerations, the two-step approach is generally preferred because of the symmetrical property of the GCC function in the time delay variable. Furthermore, the localization performance based on the optimum time delay processor was compared to the conventional approach. The result of this study showed that the optimum processor yields a one-sigma localization error ellipse which is approximately one-half smaller than the conventional approach. This improvement comes directly from a better bearing estimation. The range variance is identical between the optimum and the conventional approach.

3. A major assumption used in the derivation of the optimum multisensor, multitarget time delay processor is the known target and noise spectra. To relax this constraint, a joint time delay and spectral estimator is derived. The result of this study showed that the time delay estimate and the spectral estimate are uncorrelated. This implies that for the unknown target spectrum case, a joint time delay and spectral estimator can be implemented. The spectral estimation process does not degrade the performance of the time delay estimate. Furthermore, it was found that while the variance of the time delay estimate decreases as an inverse function of the observation time, the variance of the spectral estimate decreases as an inverse square function of the observation time.

4. The optimum multisensor, multitarget time delay processor is an order of magnitude more complex than the conventional GCC processor. Therefore, for a practical implementation, a suboptimum realization should be considered. One suboptimum procedure is to assume a low target signal-to-background noise environment. It was found that

the resulting processor is significantly simplified. From the multitarget viewpoint, the conventional GCC processor can be considered as a suboptimum processor. The performance of the GCC processor is compared to the optimum processor as a function of time delay separation.

5. Because of its simplicity, the conventional GCC processor is implemented in many existing sonar systems. On the other hand, the application of the optimum multisensor, multitarget processor requires a major modification of the conventional GCC approach. Therefore, an alternate approach is to provide additional multitarget processing at the GCC output. For this purpose, a post GCC multitarget estimator (or more appropriately the matched estimator) was investigated. In essence, the matched estimator determines the best estimate of the unknown parameter vector from a reference function which matches the observed noisy GCC output under a Least Mean Square (LMS) criteria. The matched estimator was simulated. The simulation results were compared to the theoretical predictions as well as to the optimum processor. The results of this study indicated that the matched estimator provides comparable performance with the optimum processor.

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LIST OF SYMBOLS AND ABBREVIATIONS

Symbols

a_{ij}	signal attenuation from target j to sensor i
Arg( )	Argument of
b_i^j	bias correction for optimum estimate of intersensor time delay i of target j.
B	highest frequency of either the signal or noise power spectrum
c	propagation speed of the medium
COV( )	statistical covariance
D_{ij}	propagation delay from target j to sensor i
$\underline{D}$	time delay vector
D	time delay matrix
diag( )	a diagonal matrix
E( )	statistical expectation
exp( )	exponential function
G_{kj}	effective array gain at frequency ω_k of target j
$\lvert\tilde{h}_{kj}\rvert^2$	multisensor, multitarget spectral shaping filter
J	number of target
J_{ij}	ij element of Fisher Information Matrix
k	discrete frequency index
$\underline{\ell}_i$	location vector of sensor i
log( )	natural logorithm of
M	number of sensor
M_{12}	mutual dependence
MAX( )	maximum of
$n_i(t)$	noise waveform at sensor i
N_k	discrete noise power spectrum at frequency ω_k

LIST OF SYMBOLS AND ABBREVIATIONS (Cont'd)

Symbols

$\tilde{N}_k$	multitarget equivalent noise discrete power spectrum
$\underline{0}$	a zero vector
p( )	probability density function
P_{kj}	outer product of multitarget steering vector $\underline{V}_{kj}$
Q_k	normalized discrete spatial noise covariance matrix at frequency ω_k
$\tilde{Q}_k$	equivalent noise covariance matrix
$\tilde{Q}^{-1}(\omega)$	multitarget spatial whitening matrix
$\underline{r}_j$	target j location vector
R_k	observation covariance matrix at frequency ω_k
$R_{12}(\tau)$	generalized cross correlation between channels 1 and 2
$\|R_k\|$	determinant of matrix R_k
$\|r\|$	absolute value of a scalar r
S_k	discrete signal power spectral level at frequency ω_k
t	time variable
T	basic waveform observation interval
$(\ )^T$	transpose of
tr( )	trace of a matrix
$(\ )^{-1}$	inverse of
( )*	complex conjugate transpose of
$\underline{V}_{kj}$	multisensor, multitarget steering vector of target j at frequency ω_k
V_{kj}	multisensor, multitarget steering matrix
VAR( )	statistical variance
W_k	propagation spectral matrix
$y(t, \underline{\ell}_i)$	observed waveform at sensor i

LIST OF SYMBOLS AND ABBREVIATIONS (Cont'd)

Symbols

$\underline{\alpha}_k$	Fourier coefficient vector of multisensor, multitarget observation at frequency ω_k
$\underline{\beta}_k$	Fourier coefficient vector of multitarget waveform
$\underline{\eta}_k$	Fourier coefficient vector of multisensor noise waveform
ϵ	element of the set
ω_k	discrete frequency at $\omega = 2\pi k/T$
$\underline{\theta}$	unknown parameter vector
∇	gradient operator
τ_i	intersensor time delay between sensor i + 1 and i
ρ_{ij}	correlation coefficient between τ_i and τ_j
$\rho(\tau)$	normalized correlation function
Π	series product
Σ	series sum
$\Lambda(\ )$	log-likelihood function
$\frac{\partial}{\partial \theta_i}(\ )$	partial derivative w.r.t. θ_i

Abbreviations

CRLB	Cramer-Rao Lower Bound
dB	decibel
FFT	Fast Fourier Transform
GCC	Generalized Cross Correlation
INR	Interference-to-Noise Ratio
ISR	Interference-to-Signal Ratio
LMS	Least Mean Square
ML	Maximum Likelihood

LIST OF SYMBOLS AND ABBREVIATIONS (Cont'd)

Abbreviations

MLE	Maximum Likelihood Estimator
MSC	Magnitude Square Coherence
OMP	Optimum Multitarget Processor
pdf	Probability Density Function
RMS	Rooted Mean Square
SIR	Signal-to-Interference Ratio
SNR	Signal-to-Noise Ratio
SPLOT	Stationary Parameter Long Observation Time
TDOA	Time Difference of Arrival
w.r.t.	with respect to

OPTIMUM MULTISENSOR, MULTITARGET TIME DELAY ESTIMATION

CHAPTER 1
INTRODUCTION

1.1 BACKGROUND

In passive sonar signal processing, one of the basic measured variables is the time difference of arrival (TDOA), or simply the time delay between two sensor arrays. Since the time delay vector from targets to sensors is functionally related to the target-sensor geometry, the final parameters of interest (i.e., the target location parameters) can therefore be obtained (for appropriate sensor distribution) via a direct inverse relation from measured time delays.

The measurement of time delay is mechanized through a Generalized Cross Correlation (GCC) function. The GCC is derived as a Maximum Likelihood Estimator (MLE) operating between any two sensor arrays. Detailed discussion on the subject of the GCC can be found in Knapp and Carter (Reference 1), Carter (Reference 2), Hahn and Tretter (Reference 3), and Hassab and Boucher (Reference 4).

In the above literature, the GCC is optimized under a single target assumption (or more specifically, a single coherent noise source). In the presence of multiple targets, or multipath environment, there are multiple correlation peaks. The existence of multiple correlation peaks causes performance degradation to the existing measurement system. The extent of this degradation is a function of signal spectral characteristics, signal-to-noise ratio (SNR), signal-to-interference ratio (SIR), and the relative time delay separation.

In a multisensor, multitarget environment the primary cause of performance degradation of the existing system is due to the mismatch between the signal processor design and the environment in which the signal processor must operate. Therefore, a logical approach to minimize the loss of performance is to examine the optimum structure of the signal processor under a multisensor, multitarget environment. Knowing the form of the optimum processor, one can then explore various options for a practical system realization.

1.2 TECHNICAL OBJECTIVES

The primary objective of this study is to (1) derive the optimum time delay estimator under a multisensor, multitarget environment, (2) evaluate the appropriate performance bound, and (3) investigate suboptimal processor realizations.

1.3 PREVIOUS WORK

Optimum signal processor design under a stationary multisensor, multitarget environment has been studied by a number of researchers, i.e., the work by Schweppe (Reference 5) on sensor array data processing for multiple signal source, Schultheiss (Reference 6) on passive sonar detection in the presence of interference, and Anderson and Rudnick (Reference 7) on rejection of coherent signal arrival. In addition, there are the works by Capon (Reference 8), Steinberg (Reference 9), Cox (Reference 10), McGarty (Reference 11), Rockmore and Bershad (Reference 12), and Owsley and Swope (Reference 13). With the possible exception of the work by Schweppe and Owsley/Swope, the primary efforts of these studies were on determining the effect of interference on the existing (single target) processor. Using a Least Mean Square approach, Schweppe derived a decoupled beamformer. However, his formulation ignored the effect of dependence of the covariance matrix on the unknown parameter. On the other hand, Owsley and Swope derived a single frequency multichannel focused beamformer using a Weighted Least Square approach. However, Owsley and Swope's formulation did not include the bias correction term.

1.4 TECHNICAL APPROACH AND ORGANIZATION

This study provides a fundamental examination of the optimum signal processor design for time delay estimation under the assumption of a multisensor, multitarget environment. Using an MLE procedure, an optimum multisensor, multitarget time delay estimator is derived. The resulting signal processor is reduced to its simplest form for system realization. In addition, this

study derives the appropriate performance bound for the resulting estimator. Comparisons between optimum and suboptimum realizations are also discussed.

The organization of this report is as follows: Chapter 2 discusses and formulates the multisensor, multitarget time delay estimation problem. Chapter 3 derives the optimum time delay estimator and an appropriate performance bound. In addition, the extension of time delay estimation to localization parameter estimation and power spectral estimation is considered. Chapter 4 provides some discussion on suboptimum processor realization. Chapter 5 discusses alternate approaches for improved multitarget parameter resolution. Chapter 6 presents the summary and the conclusions of the study.

CHAPTER 2
PROBLEM FORMULATION

2.1 INTRODUCTION

In this chapter we discuss the mathematical description of the sensor observations and formulate the general multisensor, multitarget time delay estimation problem. The optimum signal processor is sought via an MLE procedure. In general, we assume the target/sensor environment consists of M sensors and J targets (or J arrival paths). Target waveforms are assumed mutually uncorrelated. As an introduction to the general multitarget, multisensor problem formulation, we first consider the basic description of observables for a single target case.

2.2 DESCRIPTION OF OBSERVABLES

Let M sensor arrays be distributed arbitrarily in space. These M sensor arrays then produce M continuous waveforms which represent the M spatial samples of a random field generated by the target signal in additive ambient noise. Let the M waveforms be observed for a duration of T seconds and let the received waveforms be written as:

$$y(t, \underline{\ell}_i) = a_i \, s_i(t) + n_i(t) \; ; \qquad \begin{array}{l} i = 1, 2, \ldots, M \\ t \in [0, T] \end{array} \qquad (2.2\text{-}1)$$

where a_i, $s_i(t)$, and $n_i(t)$ are the signal attenuation factor, signal waveform, and the noise waveform, respectively, for the ith sensor. We note that the observed waveform $y(t, \underline{\ell}_i)$ is a random function of space and time, and $\underline{\ell}_i$ is the location vector of the ith sensor with respect to a known reference point. (See Figure 2-1.)

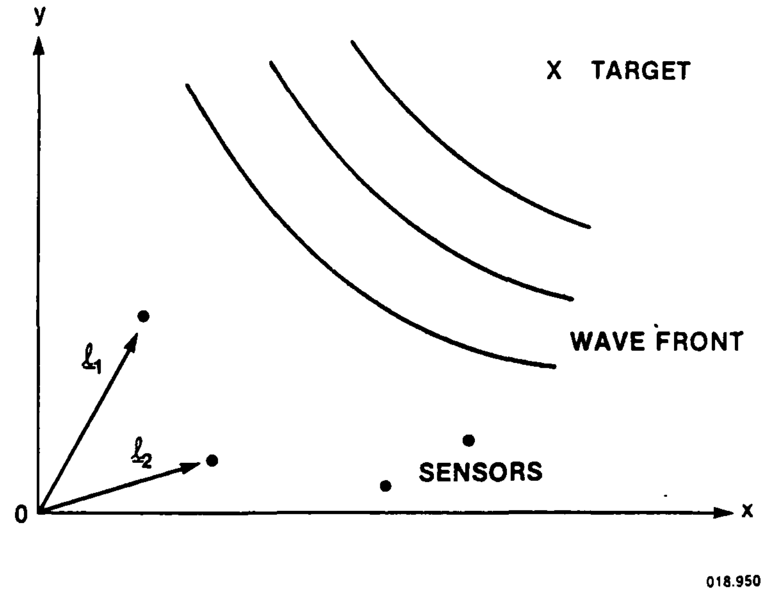

Figure 2-1. Description of Passive Sensor Observation

To simplify the analysis, the following assumptions have been made:

1. Pure Time Delay Channel - The signal waveforms received at each sensor are identical except for a pure time delay. Thus, the signal waveform (for the single target case) of the ith sensor can be written as $s_i(t) = s(t + D_i)$, where D_i is the propagation delay from the signal source to the ith sensor. Therefore, Equation (2.2-1) can be rewritten as:

$$y(t, \underline{\ell}_i) = a_i\, s(t + D_i) + n_i(t)\ ; \qquad \begin{array}{l} i = 1, 2, \ldots, M \\ t \in [0, T]\ . \end{array} \tag{2.2-2}$$

From Figure 2-2, we see that the propagation time delay is given by the relation

$$D_i = \frac{|\underline{r} - \underline{\ell}_i|}{c} \tag{2.2-3}$$

where

$\underline{r}$ = position vector of the signal source

$\underline{\ell}_i$ = location vector of the ith sensor

c = propagation speed of the medium.

2. Noise at each sensor is assumed additive with known spatial and time correlation function.

3. The observation time T is much longer than the travel time of the wavefront across the sensor array.

4. Both the signal process and the noise process are mutually uncorrelated, zero mean, Gaussian, stationary in time, homogenous in space (Yaglom (Reference 14)), and have a known band-limited power spectrum.

We note that the set of observations $y(t, \underline{\ell}_i)$; i = 1, 2, ..., M; $t \in [0, T]$ contains a complete description of the observables. However, it has infinite dimensions and is analytically intractable. Therefore, we wish to

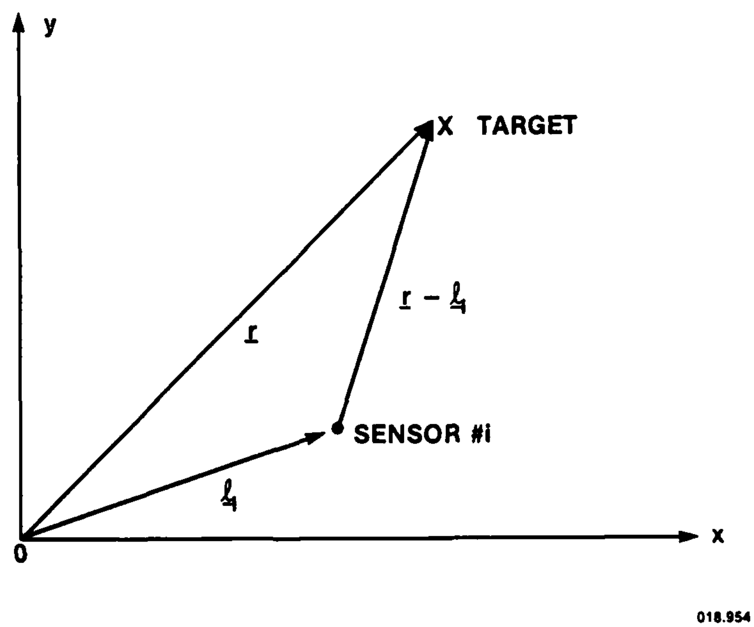

Figure 2-2. Target-to-Sensor Geometry

represent the time-limited observation of the waveform by a finite set of discrete random variables. Because of the time delays in the signal waveform, a convenient approach is to represent the observation in terms of a Fourier series expansion as discussed by MacDonald and Schultheiss (Reference 15), Knapp and Carter (Reference 1), and Rockmore (Reference 16). The resulting coefficients are almost uncorrelated for a large bandwidth time product as demonstrated by Hodgkiss and Nolte (Reference 17). Therefore, Equation (2.2-2) can be represented by a Fourier coefficient vector as:

$$\underline{\alpha}_k = \beta_k \, \underline{V}_k + \underline{n}_k \; ; \qquad k = \pm 1, \pm 2, \ldots \tag{2.2-4}$$

where

$$\underline{\alpha}_k = (\alpha_{1k} \; \alpha_{2k} \cdots \alpha_{Mk})^T \tag{2.2-5a}$$

$$\underline{V}_k = \left(a_1 e^{j\omega_k D_1} \; a_2 e^{j\omega_k D_2} \; \ldots \; a_M e^{j\omega_k D_M} \right)^T \tag{2.2-5b}$$

$$\underline{n}_k = (n_{1k} \; n_{2k} \cdots n_{Mk})^T \tag{2.2-5c}$$

are the vectors of observation, signal steering, and additive noise, respectively, for all sensor outputs at frequency $\omega_k = 2\pi k/T$. Furthermore, the variables α_{ik}, β_k and n_{ki} are defined as follows:

$$\alpha_{ik} = \frac{1}{T} \int_0^T y(t, \underline{\ell}_i) \, e^{-j\omega_k t} \, dt \tag{2.2-6a}$$

$$\beta_k = \frac{1}{T} \int_0^T s(t)\, e^{-j\omega_k t}\, dt \qquad (2.2\text{-}6b)$$

$$n_{ik} = \frac{1}{T} \int_0^T n_i(t)\, e^{-j\omega_k t}\, dt \; . \qquad (2.2\text{-}6c)$$

Note that for real signal and noise waveforms, the Fourier coefficients are conjugate symmetric; i.e., $\alpha_{i,-k} = \alpha^*_{ik}$. Thus, for a band-limited process one only needs to consider frequency components k = 1, 2, ..., B, where B is the highest cut-off frequency of either the signal or the noise spectrum. For a low pass power spectrum, B can be identified as the one-sided bandwidth.

Since both the signal and noise are zero mean Gaussian processes, it is easily established that $\underline{\alpha}_k$ is a zero mean Gaussian random vector with covariance matrix:

$$R_k = E(\underline{\alpha}_k\, \underline{\alpha}_k^*)$$

$$= S_k\, \underline{V}_k\, \underline{V}_k^* + N_k\, Q_k \qquad (2.2\text{-}7a)$$

where $\underline{\alpha}_k^*$ is the complex conjugate transpose of $\underline{\alpha}_k$; S_k and N_k are the discrete signal and noise power spectral density, respectively, at frequency ω_k; and Q_k is the normalized spatial covariance matrix of the noise vector such that

$$\mathrm{tr}(Q_k) = M \qquad (2.2\text{-}7b)$$

where tr( ) defines the trace of a matrix.

Now writing

$$\underline{\alpha} = (\underline{\alpha}_1^T \ \underline{\alpha}_2^T \ \cdots \ \underline{\alpha}_B^T)^T$$

as an MB dimensional complex column observation vector, we obtain tne first and the second order moments of $\underline{\alpha}$ as:

$$E(\underline{\alpha}) = \underline{0} \tag{2.2-8a}$$

$$E(\underline{\alpha}\ \underline{\alpha}^*) = \text{diag}\{R_k\} \tag{2.2-8b}$$

which is a block diagonal since for a large bandwidth-time product,

$$E(\underline{\alpha}_k\ \underline{\alpha}_\ell^*) \approx 0 \quad \text{for } k \neq \ell\ .$$

Now let

$$\underline{D} = (D_1\ D_2\ \cdots\ D_M)^T$$

be the propagation delay vector, then the probability density function (pdf) of $\underline{\alpha}_k$ conditioned on $\underline{D}$ is complex Gaussian (Appendix A) given by

$$p(\underline{\alpha}_k|\underline{D}) = \pi^{-M}|R_k|^{-1} \exp\{-\underline{\alpha}_k^* R_k^{-1} \underline{\alpha}_k\} \tag{2.2-9}$$

and since $\underline{\alpha}_k$ and $\underline{\alpha}_\ell$ are uncorrelated for $k \neq \ell$, they are also independent. Hence, one can write

$$p(\underline{\alpha}|\underline{D}) = p(\underline{\alpha}_1, \underline{\alpha}_2, \ldots, \underline{\alpha}_B|\underline{D})$$

$$= \prod_{k=1}^{B} p(\underline{\alpha}_k|\underline{D})$$

$$= \pi^{-MB} \prod_{k=1}^{B} |R_k|^{-1} \exp\{-\underline{\alpha}_k^* R_k^{-1} \underline{\alpha}_k\} \tag{2.2-10}$$

as the pdf of the complete observation vector $\underline{\alpha}$ conditioned on the propagation time delay vector $\underline{D}$. Thus Equation (2.2-10) provides a complete statistical description of the observation conditioned on the unknown time delay vector.

2.3 STATEMENT OF THE PROBLEM

In this section we formulate the general multisensor, multitarget time delay estimation problem in terms of the observation vector described in the previous section. We consider the problem of optimum time delay estimation from M sensor arrays in the presence of J possible targets. The final objective of sensor array time delay estimation is to provide optimum estimates of target localization parameters (i.e., range and bearing). We assume that interarray separation is large compared to the size of each sensor array. Such an arrangement is normally found in large aperture surveillance systems. See, for example, the large seismic array described by Capon (Reference 18). A recent study by Carter (Reference 2) has shown that for passive localization of an acoustic source, the variance of the bearing error is inversely proportional to the square of the base length while the variance of the range error is proportional to the fourth-power of the ratio of the true range to the base length. Thus, a long base length is desirable in reducing the variance of the estimates.

Let the M sensor array outputs from J acoustic sources be written as:

$$y(t, \underline{\ell}_i) = \sum_{j=1}^{J} a_{ij}\, s_j(t + D_{ij}) + n_i(t) \; ; \qquad t \in [0, T] \tag{2.3-1}$$

where $i = 1, 2, \ldots, M$; a_{ij} is the signal attenuation factor for the jth target to the ith sensor; D_{ij} is the propagation time delay from target j to sensor i; and $s_j(t)$ and $n_i(t)$ are the band-limited signal and noise processes with the usual assumptions of zero mean, Gaussian, time stationary, and spatially homogeneous. Fourier expansion of Equation (2.3-1) yields the finite dimensional frequency domain representation as:

$$\underline{\alpha}_k = W_k \underline{\beta}_k + \underline{n}_k \; ; \qquad k = 1, 2, \ldots, B \tag{2.3-2a}$$

where

$$\underline{\alpha}_k = (\alpha_{1k} \; \alpha_{2k} \cdots \alpha_{Mk})^T \tag{2.3-2b}$$

$$\underline{\beta}_k = (\beta_{k1} \; \beta_{k2} \cdots \beta_{kJ})^T \tag{2.3-2c}$$

$$\underline{n}_k = (n_{1k} \; n_{2k} \cdots n_{Mk})^T \tag{2.3-2d}$$

and

$$W_k = \left(a_{ij} e^{j\omega_k D_{ij}} \right) ; \qquad \begin{array}{l} i = 1, 2, \ldots, M \\ j = 1, 2, \ldots, J \end{array} \tag{2.3-2e}$$

is a M x J complex "propagation delay" matrix, where M is the number of sensors, J is the number of targets, and $\omega_k = 2\pi k/T$ is the discrete frequency. Similar to the development in the last section, the observation vector $\underline{\alpha}_k$ is Gaussian with the following statistics:

$$E(\underline{\alpha}_k) = \underline{0} \tag{2.3-3a}$$

$$E(\underline{\alpha}_k \, \underline{\alpha}_\ell^*) = \begin{cases} C; \text{ if } k \neq \ell \\ R_k; \text{ if } k = \ell \end{cases} \tag{2.3-3b}$$

where

$$R_k \triangleq W_k\, S_k\, W_k^* + N_k\, Q_k \tag{2.3-3c}$$

and

$$S_k = E(\underline{\beta}_k\, \underline{\beta}_k^*) \ . \tag{2.3-3d}$$

For uncorrelated sources, we have

$$S_k = \mathrm{diag}\{S_{k1},\, S_{k2},\, \cdots ,\, S_{kJ}\} \tag{2.3-3e}$$

as a J x J diagonal matrix with element S_{kj} as the signal power at frequency ω_k of target j. Thus, the pdf of $\underline{\alpha}_k$ conditioned on the time delay matrix defined by

$$D = \{D_{ij}\} \ ; \qquad \begin{array}{l} i = 1,\, 2,\, \ldots,\, M \\ j = 1,\, 2,\, \ldots,\, J \end{array} \tag{2.3-3f}$$

is

$$p(\underline{\alpha}_k|D) = \pi^{-M}\, |R_k|^{-1} \exp\{-\underline{\alpha}_k^*\, R_k^{-1}\, \underline{\alpha}_k\} \ . \tag{2.3-4}$$

Finally, writing

$$\underline{\alpha} = (\underline{\alpha}_1^T\, \underline{\alpha}_2^T \cdots \underline{\alpha}_B^T)^T \ , \tag{2.3-5a}$$

then the pdf of $\underline{\alpha}$, the complete observation vector, conditioned on D, the multisensor, multitarget time delay matrix, can be written as

$$p(\underline{\alpha}|D) = \prod_{k=1}^{B} p(\underline{\alpha}_k|D)$$

$$= \pi^{-MB} \prod_{k=1}^{B} |R_k|^{-1} \exp\{-\underline{\alpha}_k^*\, R_k^{-1}\, \underline{\alpha}_k\} \ . \tag{2.3-5b}$$

Equation (2.3-5b) constitutes the basis for the derivation of the multisensor, multitarget optimum time delay processor. An MLE is obtained by maximizing the conditional pdf with respect to each element of the propagation time delay matrix. Thus, symbolically one can write

$$\hat{D}_{ML} = \underset{D}{\text{Max Arg}}\ p(\underline{\alpha}|D) \ . \tag{2.3-6}$$

CHAPTER 3
OPTIMUM MULTITARGET PARAMETER ESTIMATION

3.1 INTRODUCTION

In principle the optimum estimate of the time delay matrix D is obtained by solving Equation (2.3-6) numerically. In general, a direct implementation of Equation (2.3-6) results in a very complex processor. However, without loss of performance, Equation (2.3-6) can be reduced to its simplest form by mathematical manipulations. The resulting processor is usually realizable.

In this chapter we investigate the fine structure of the optimum multi-target time delay signal processor. Since time delays are modeled as unknown constants, we seek an optimum estimator via an MLE approach. We provide a performance bound for the resulting estimator. We study in detail the two-sensor, two-target case. For the three-sensor case we establish the relationship between optimum localization parameter estimation and optimum time delay estimation. Finally, we investgate the structure of the optimum power spectral estimator when the target signal power spectrum is not known a priori.

3.2 THE LIKELIHOOD EQUATION

It was shown in Chapter 2, Equation (2.3-5b), that the pdf of the sensor observation vector $\underline{\alpha}$ conditioned on the time delay matrix D is given by

$$p(\underline{\alpha}|D) = \prod_{k=1}^{B} p(\underline{\alpha}_k|D) \tag{3.2-1a}$$

where

$$p(\underline{\alpha}_k|D) = \pi^{-M}|R_k|^{-1} \exp\{-\underline{\alpha}_k^* R_k^{-1} \underline{\alpha}_k\} \ . \tag{3.2-1b}$$

The ij element of the time delay matrix D is given by (see Equation (2.3-3f))

$$D_{ij} = \frac{|\underline{r}_j - \underline{\ell}_i|}{c} ; \qquad \begin{array}{l} i = 1, 2, \ldots, M \\ j = 1, 2, \ldots, J \end{array} \tag{3.2-2}$$

where $\underline{r}_j$ and $\underline{\ell}_i$ denote the vector locations of target j and sensor i, respectively. The log-likelihood function of Equation (3.2-1a) is defined by

$$\begin{aligned} \Lambda(D) &= \sum_{k=1}^{B} \log p(\underline{\alpha}_k | D) \\ &= -MB \log(\pi) - \sum_{k=1}^{B} \Lambda_k(D) \end{aligned} \tag{3.2-3a}$$

where

$$\Lambda_k(D) = \log|R_k| + \underline{\alpha}_k^* R_k^{-1} \underline{\alpha}_k \tag{3.2-3b}$$

and R_k is the covariance matrix of the zero mean complex Gaussian vector $\underline{\alpha}_k$ given by (see Equation (2.3-3c))

$$R_k = W_k S_k W_k^* + N_k Q_k \tag{3.2-3c}$$

where for the uncorrelated source case,

$$S_k = \mathrm{diag}\{S_{k1}, S_{k2}, \ldots, S_{kJ}\} \tag{3.2-3d}$$

is a J x J diagonal signal power density matrix whose jth diagonal entry denotes the discrete signal power density of target j at frequency ω_k. N_k is

the discrete noise power density, and Q_k is the normalized covariance matrix. Finally, the M x J "propagation delay matrix", W_k, is defined by

$$W_k = \left(a_{ij} e^{j\omega_k D_{ij}} \right) ; \qquad \begin{array}{l} i = 1, 2, \ldots, M \\ j = 1, 2, \ldots, J \end{array} \tag{3.2-3e}$$

where a_{ij} denotes the signal attenuation factor from target j to sensor i. Now writing

$$W_k = (\underline{\tilde{V}}_{k1}\ \underline{\tilde{V}}_{k2} \cdots \underline{\tilde{V}}_{kJ}) \tag{3.2-4a}$$

where

$$\underline{\tilde{V}}_{kj} = \left(a_{1j} e^{j\omega_k D_{1j}} \quad a_{2j} e^{j\omega_k D_{2j}} \cdots a_{Mj} e^{j\omega_k D_{Mj}} \right)^T \tag{3.2-4b}$$

is the attenuated signal steering vector of target j at frequency ω_k. Using the fact that S_k is a diagonal matrix for uncorrelated sources, we have the obvious identity:

$$W_k\ S_k\ W_k^* = \sum_{j=1}^{J} S_{kj}\ \underline{\tilde{V}}_{kj}\ \underline{\tilde{V}}_{kj}^*$$

$$= \sum_{j=1}^{J} S_{kj}\ P_{kj} \tag{3.2-4c}$$

where P_{kj} is defined by

$$P_{kj} \overset{\Delta}{=} \underline{\tilde{V}}_{kj}\ \underline{\tilde{V}}_{kj}^*$$

$$= \underline{V}_{kj}\ \underline{V}_{kj}^* \ .$$

Note that we have chosen to let

$$\underline{\tilde{V}}_{kj} = e^{j\omega_k D_{1j}} \underline{V}_{kj} \tag{3.2-4d}$$

or equivalently (to reference all time delays to the first sensor), one can write

$$\underline{V}_{kj} = \left(a_{1j}\ a_{2j}e^{j\omega_k \Delta_{1j}} \ \ldots\ a_{Mj}e^{j\omega_k \Delta_{M-1,j}}\right)^T \tag{3.2-4e}$$

where Δ_{ij} is given by the relation

$$\Delta_{ij} = D_{i+1,j} - D_{1j}\ ; \qquad i = 1, 2, \ldots, M-1\ . \tag{3.2-4f}$$

Now P_{kj} is an M x M Hermitian matrix of rank one and is a function of the attenuation vector $\underline{a}_j$ and the time delay vector $\underline{\Delta}_j$. The vectors $\underline{a}_j$ and $\underline{\Delta}_j$ are given by

$$\underline{a}_j = (a_{1j}\ a_{2j}\ \cdots\ a_{Mj})^T \tag{3.2-4g}$$

and

$$\underline{\Delta}_j = (\Delta_{1j}\ \Delta_{2j}\ \cdots\ \Delta_{M-1,j})^T\ . \tag{3.2-4h}$$

Note that given M sensors, there are M-1 independent time delay pairs from a possible total of M(M-1)/2. The selection of this set is not unique. However, for time delay estimation, it is reasonable to assume a set with minimum total delay. This set (for a line array) is given by the inter-sensor time delay vector:

$$\underline{\tau}_j = (\tau_{1j}\ \tau_{2j}\ \cdots\ \tau_{M-1,j})^T\ ; \qquad j=1, 2, \ldots, J \tag{3.2-4i}$$

where

$$\tau_{ij} = D_{i+1,j} - D_{ij}\ .$$

Note that the Δ_{ij}'s can be expressed in terms of the τ_{ij}'s. For example:

$$\Delta_{ij} = D_{i+1,j} - D_{1j}$$

$$= \sum_{n=1}^{i} D_{n+1,j} - D_{nj}$$

$$= \sum_{n=1}^{i} \tau_{nj}$$

$$= \underline{U}_i^T \underline{\tau}_j \tag{3.2-4j}$$

where $\underline{U}_i$ is a column vector whose first i entries are one and the remainder are zero.

Furthermore, by writing the mn element of the matrix P_{kj} as P_{kj}^{mn}, and from Equation (3.2-4e), we can identify the relation

$$P_{kj}^{mn} = a_{mj}\, a_{nj}\, e^{j\omega_k(\Delta_{m-1,j} - \Delta_{n-1,j})} \;; \qquad \begin{array}{l} m = 1, 2, \ldots, M \\ n = 1, 2, \ldots, M\,. \end{array} \tag{3.2-5a}$$

Now utilizing the relation in Equation (3.2-4j), we obtain

$$P_{kj}^{mn} = a_{mj}\, a_{nj}\, e^{j\omega_k(\underline{U}_{m-1} - \underline{U}_{n-1})^T \underline{\tau}_j}\,. \tag{3.2-5b}$$

Thus we have shown that the observation covariance matrix can be expressed as a function of inter-sensor time delays instead of the actual propagation delays. Therefore, the direct observable quantities are time delays instead of the propagation delays.

Using Equation (3.2-4c) in (3.2-3c), one obtains an alternate expression for the observation covariance matrix:

$$R_k = \sum_{j=1}^{J} S_{kj}\ P_{kj} + N_k\ Q_k\ . \tag{3.2-6}$$

Equation (3.2-6) can also be summed in the following way:

$$\begin{aligned} R_k &= S_{k1}\ P_{k1} + \tilde{N}_{k1}\ \tilde{Q}_{k1} \\ &= S_{k2}\ P_{k2} + \tilde{N}_{k2}\ \tilde{Q}_{k2} \\ &\quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ &\quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ &\quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ &= S_{kJ}\ P_{kJ} + \tilde{N}_{kJ}\ \tilde{Q}_{kJ} \end{aligned} \tag{3.2-7a}$$

where

$$\tilde{N}_{kj} = \sum_{\substack{i=1 \\ i \neq j}}^{J} S_{ki} + N_k\ ; \qquad j = 1, 2, \ldots, J \tag{3.2-7b}$$

and

$$\tilde{Q}_{kj} = \left(\sum_{\substack{i=1 \\ i \neq j}}^{J} S_{ki}\ P_{ki} + N_k\ Q_k \right) / \tilde{N}_{kj}\ . \tag{3.2-7c}$$

Note that $\tilde{Q}_{kj}$ can be regarded as an equivalent noise process and is independent of the parameter $\underline{\tau}_j$. For simplicity we shall assume the propagation attenuation coefficients are known and, for convenience, assume they are equal to unity.

Finally, writing the incremental (intersensor) time delay vector for all targets as

$$\underline{\theta} = (\underline{\tau}_1^T\ \underline{\tau}_2^T \ldots \underline{\tau}_J^T)^T\ , \tag{3.2-8}$$

a p-parameter column vector where $p = J(M-1)$, the vector log-likelihood function (Equations (3.2-3a) and (3.2-3b)) becomes

$$\Lambda(\underline{\theta}) = -MB \log(\pi) - \sum_{k=1}^{B} \Lambda_k(\underline{\theta}) \tag{3.2-9}$$

where

$$\Lambda_k(\underline{\theta}) = \log|R_k| + \underline{\alpha}_k^* R_k^{-1} \underline{\alpha}_k \, . \tag{3.2-10}$$

The MLE of $\underline{\theta}$ is one that maximizes the likelihood function $\Lambda(\underline{\theta})$. A necessary condition for the location of the maximum is given by the vector likelihood equation

$$\nabla\Lambda(\underline{\theta}) = -\sum_{k=1}^{B} \nabla\Lambda_k(\underline{\theta}) = \underline{0} \tag{3.2-11}$$

where ∇ is the gradient operator given by the column vector

$$\nabla = \left(\frac{\partial}{\partial\theta_1} \frac{\partial}{\partial\theta_2} \cdots \frac{\partial}{\partial\theta_p} \right)^T . \tag{3.2-12}$$

3.3 MULTIPLE PARAMETER ESTIMATION

It was shown in the previous section that the optimal estimate of the time delay vector is hinged on solving the vector likelihood equation. For the MLE, we have

$$\underline{f}(\underline{\theta}) \triangleq \nabla\Lambda(\underline{\theta}) = -\sum_{k=1}^{B} \nabla\Lambda_k(\underline{\theta}) = \underline{0} \tag{3.3-1}$$

where $\underline{f}(\underline{\theta})$ is a vector function. The likelihood function $\Lambda_k(\underline{\theta})$ is defined as in Equation (3.2-10).

In this section we examine the solution of the likelihood equation in general and in Section 3.5 we explore in detail the structure of the optimum multitarget, multisensor processor.

Let θ_j denote the jth element of the time delay vector $\underline{\theta}$, then Equation (3.3-1) can be written as

$$\frac{\partial}{\partial\theta_j} \Lambda(\underline{\theta}) = - \sum_{k=1}^{B} \frac{\partial}{\partial\theta_j} \Lambda_k(\underline{\theta}) = 0 \; ; \qquad j = 1, 2, \ldots, p \, . \tag{3.3-2}$$

From Equation (3.2-10) we obtain

$$\frac{\partial}{\partial\theta_j} \Lambda_k(\underline{\theta}) = |R_k|^{-1} \frac{\partial}{\partial\theta_j} |R_k| + \underline{\alpha}_k^* \frac{\partial R_k^{-1}}{\partial\theta_j} \underline{\alpha}_k$$

$$= \mathrm{tr}\left(R_k^{-1} \frac{\partial R_k}{\partial\theta_j} \right) + \underline{\alpha}_k^* \frac{\partial R_k^{-1}}{\partial\theta_j} \underline{\alpha}_k \tag{3.3-3}$$

where tr( ) denotes the trace of a matrix. Note that the evaluation of the first term is a straightforward application of the chain rule. The exact procedure can be found in Rockmore (Reference 16). The derivative of Equation (3.3-3) w.r.t. θ_i is

$$\frac{\partial^2 \Lambda_k}{\partial\theta_i \partial\theta_j} (\underline{\theta}) = \mathrm{tr}\left(\frac{\partial R_k^{-1}}{\partial\theta_i} \frac{\partial R_k}{\partial\theta_j} + R_k^{-1} \frac{\partial^2 R_k}{\partial\theta_i \partial\theta_j} \right) + \underline{\alpha}_k^* \frac{\partial^2 R_k^{-1}}{\partial\theta_i \partial\theta_j} \underline{\alpha}_k \, . \tag{3.3-4}$$

Using the linear property of the trace operator and the relation

$$\frac{\partial R_k^{-1}}{\partial\theta_i} = -R_k^{-1} \frac{\partial R_k}{\partial\theta_i} R_k^{-1} \, ,$$

Equation (3.3-4) becomes

$$\frac{\partial^2 \Lambda_k(\underline{\theta})}{\partial\theta_i \partial\theta_j} = -\mathrm{tr}\left(R_k^{-1} \frac{\partial R_k}{\partial\theta_i} R_k^{-1} \frac{\partial R_k}{\partial\theta_j}\right) + \mathrm{tr}\left(R_k^{-1} \frac{\partial^2 R_k}{\partial\theta_i \partial\theta_j}\right) + \underline{\alpha}_k^* \frac{\partial^2 R_k^{-1}}{\partial\theta_i \partial\theta_j} \underline{\alpha}_k . \tag{3.3-5}$$

Equation (3.3-5) is important in evaluating the performance bound of an estimator. Substituting Equation (3.3-3) in (Equation 3.3-1), one obtains the set of necessary conditions for the ML estimate:

$$f_i(\underline{\theta}) = -\sum_{k=1}^{B} \left[\mathrm{tr}\left(R_k^{-1} \frac{\partial R_k}{\partial\theta_i}\right) + \underline{\alpha}_k^* \frac{\partial R_k^{-1}}{\partial\theta_i} \underline{\alpha}_k\right] = 0 ; \qquad i = 1, 2, \ldots, p. \tag{3.3-6}$$

Equation (3.3-6) is usually non-linear. The structure of the optimum processor can be found by reducing the required mathematical operations to simplest form. In principle, Equation (3.3-6) can be solved by searching the p-parameter space for a simultaneous null. A more efficient algorithm, however, is implementing a closed-loop null tracker. Further discussion on this important subject is beyond the scope of this study.

We remark that Equation (3.3-6) is the necessary condition for the existence of a maximum. For sufficiency it requires not only the condition given in Equation (3.3-6) but also the following (Bryson and Ho, Reference 19):

$$\frac{\partial}{\partial\underline{\theta}} \underline{f}(\underline{\theta}) = \frac{\partial^2 \Lambda(\underline{\theta})}{\partial\underline{\theta}^2} < 0 . \tag{3.3-7}$$

By this we mean the square matrix $\partial^2\Lambda(\theta)/\partial\underline{\theta}^2$ must be negative definite.

3.4 ESTIMATOR PERFORMANCE EVALUATION

In this section we derive the multi-parameter Cramer-Rao Lower Bound (CRLB) and show that the resulting estimate obtained from solving Equation (3.3-6) satisfies the bound for a large observation time. Therefore, the resulting estimate is efficient.

The CRLB for an unbiased estimate of the ith parameter is given by Van Trees (Reference 20):

$$\mathrm{VAR}(\hat{\theta}_i)_{ML} \geq (J^{-1})_{ii} \tag{3.4-1}$$

where $(\ )_{ii}$ denotes the ith diagonal element of a matrix and J is the Fisher Information Matrix whose ij element is defined by

$$J_{ij} \stackrel{\Delta}{=} -E\left(\frac{\partial^2 \Lambda(\theta)}{\partial\theta_i \partial\theta_j}\right) \quad ; \qquad \begin{array}{l} i = 1, 2, \ldots, P \\ j = 1, 2, \ldots, P \end{array} \tag{3.4-2}$$

where $\Lambda(\underline{\theta})$ is the log-likelihood function. Using Equations (3.3-2) and (3.3-5) in (3.4-2), we immediately obtain

$$J_{ij} \stackrel{\Delta}{=} -\sum_{k=1}^{B} \left[\mathrm{tr}\left(R_k^{-1} \frac{\partial R_k}{\partial\theta_i} R_k^{-1} \frac{\partial R_k}{\partial\theta_j}\right) - \mathrm{tr}\left(R_k^{-1} \frac{\partial^2 R_k}{\partial\theta_i \partial\theta_j} + \frac{\partial^2 R_k^{-1}}{\partial\theta_i \partial\theta_j} R_k\right)\right]\Bigg|_{\underline{\theta} = \underline{\theta}_o} \tag{3.4-3}$$

where $\underline{\theta}_o$ denotes the true parameter value. Equation (3.4-3) can be simplified as follows: Taking the derivative of the identity $R_k^{-1} R_k = I$ first w.r.t. θ_j and then w.r.t. θ_i, one obtains the relation

$$\left(R_k^{-1} \frac{\partial^2 R_k}{\partial\theta_i \partial\theta_j} + \frac{\partial^2 R_k^{-1}}{\partial\theta_i \partial\theta_j} R_k\right) = -\left(\frac{\partial R_k^{-1}}{\partial\theta_j} \frac{\partial R_k}{\partial\theta_i} + \frac{\partial R_k^{-1}}{\partial\theta_i} \frac{\partial R_k}{\partial\theta_j}\right) \quad . \tag{3.4-4}$$

Substituting Equation (3.4-4) in Equation (3.4-3), the latter can be simplified to

$$J_{ij} = \sum_{k=1}^{B} \mathrm{tr}\left(R_k^{-1} \frac{\partial R_k}{\partial \theta_j} R_k^{-1} \frac{\partial R_k}{\partial \theta_i}\right)\Bigg|_{\underline{\theta} = \underline{\theta}_0} \tag{3.4-5a}$$

$$= \sum_{k=1}^{B} \mathrm{tr}\left(- \frac{\partial R_k^{-1}}{\partial \theta_j} \frac{\partial R_k}{\partial \theta_i}\right)\Bigg|_{\underline{\theta} = \underline{\theta}_0} \tag{3.4-5b}$$

Note that because tr(AB) = tr(BA), we have $J_{ij} = J_{ji}$. Equation (3.4-5b) agrees with the expression obtained by Bangs (Reference 21). However, the derivation presented here is somewhat simpler and more direct.

Next we proceed to show that the MLE, $\hat{\underline{\theta}}_{ML}$, of $\underline{\theta}$ obtained by simultaneously solving the set of equations in Equation (3.3-6) is unbiased and achieves the CRLB.

Let $\underline{f}(\underline{\theta}) = (f_1(\underline{\theta})\ f_2(\underline{\theta})\ \ldots\ f_p(\underline{\theta}))^T$

and write the Taylor series expansion of $\underline{f}(\underline{\theta})$ about the true parameter value $\underline{\theta}_0$ as follows:

$$\underline{f}(\underline{\theta}) = \underline{f}(\underline{\theta}_0) + \frac{\partial f(\theta)}{\partial \underline{\theta}}\Bigg|_{\underline{\theta} = \underline{\theta}_0} (\underline{\theta} - \underline{\theta})_0 + \ldots . \tag{3.4-6}$$

Since $\underline{f}(\hat{\underline{\theta}}_{ML}) = 0$ by definition, we have after neglecting higher order terms (small random error assumption):

$$-\underline{f}(\underline{\theta}_0) = \frac{\partial \underline{f}(\underline{\theta}_0)}{\partial \underline{\theta}} (\hat{\theta}_{ML} - \underline{\theta}_0) . \tag{3.4-7}$$

Now assume the law of large number applies so that (for a sufficiently long observation time) one can replace the derivative by its expected value. Taking the expected value on both sides of Equation (3.4-7), one obtains

$$E(\hat{\underline{\theta}}_{ML}) = \underline{\theta}_0 - \left\{ E\left[\frac{\partial \underline{f}(\underline{\theta}_0)}{\partial \underline{\theta}}\right]\right\}^{-1} E[\underline{f}(\underline{\theta}_0)] \ . \tag{3.4-8}$$

But from Equation (3.3-6)

$$E[f_i(\underline{\theta}_0)] = -\sum_{k=1}^{B} \left[tr\left(R_k^{-1} \frac{\partial R_k}{\partial \theta_i}\right) + tr\left(\frac{\partial R_k^{-1}}{\partial \theta_i} R_k\right)\right]\Bigg|_{\underline{\theta} = \underline{\theta}_0}$$

$$= -\sum_{k=1}^{B} \left[tr\left(R_k^{-1} \frac{\partial R_k}{\partial \theta_i} + \frac{\partial R_k^{-1}}{\partial \theta_i} R_k\right)\right]\Bigg|_{\underline{\theta} = \underline{\theta}_0}$$

$$= 0 \ . \tag{3.4-9}$$

Therefore, $E(\hat{\underline{\theta}}_{ML}) = \underline{\theta}_0$, and $\hat{\underline{\theta}}_{ML}$ is an unbiased estimator. Post multipling Equation (3.4-7) by its conjugate transpose and taking the expectation yields

$$E[\underline{f}(\underline{\theta}_0)\underline{f}^*(\underline{\theta}_0)] = E\left[\frac{\partial \underline{f}(\underline{\theta}_0)}{\partial \underline{\theta}}\right] E\left[(\hat{\underline{\theta}}_{ML} - \underline{\theta}_0)(\hat{\underline{\theta}}_{ML} - \underline{\theta}_0)^T\right] E\left[\frac{\partial \underline{f}(\underline{\theta}_0)^*}{\partial \underline{\theta}}\right] \ . \tag{3.4-10}$$

Now recall that $\underline{f}(\underline{\theta}) = \nabla\Lambda(\underline{\theta})$, so one can write

$$E\left[\frac{\partial \underline{f}(\underline{\theta}_0)}{\partial \underline{\theta}}\right] = E\left[\frac{\partial^2 \Lambda(\underline{\theta})}{\partial \underline{\theta}^2}\right]\Bigg|_{\underline{\theta} = \underline{\theta}_0}$$

$$= E \begin{bmatrix} \frac{\partial^2 \Lambda(\theta)}{\partial \theta_1{}^2} & \frac{\partial^2 \Lambda(\theta)}{\partial \theta_1 \partial \theta_2} & \cdots\cdots\cdots & \frac{\partial^2 \Lambda(\theta)}{\partial \theta_1 \partial \theta_p} \\ \frac{\partial^2 \Lambda(\theta)}{\partial \theta_2 \partial \theta_1} & \frac{\partial^2 \Lambda(\theta)}{\partial \theta_2^2} & \cdots\cdots\cdots & \frac{\partial^2 \Lambda(\theta)}{\partial \theta_2 \partial \theta_p} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 \Lambda(\theta)}{\partial \theta_p \partial \theta_1} & \frac{\partial^2 \Lambda(\theta)}{\partial \theta_p \partial \theta_2} & \cdots\cdots\cdots & \frac{\partial^2 \Lambda(\theta)}{\partial \theta_p^2} \end{bmatrix}_{\theta = \underline{\theta}_0} \tag{3.4-11}$$

$$= -J \ .$$

Furthermore, we have the relation

$$E\left[\underline{f}(\underline{\theta}_0)\ \underline{f}^*(\underline{\theta}_0)\right] = E\left[\nabla\Lambda(\underline{\theta})\ \nabla^T\Lambda(\underline{\theta})\right]_{\underline{\theta} = \underline{\theta}_0} \tag{3.4-12}$$

$$= J \ .$$

Therefore, substituting Equations (3.4-11) and (3.4-12) in (3.4-10) yields

$$E\left[(\hat{\underline{\theta}}_{ML} - \underline{\theta}_0)\ (\hat{\underline{\theta}}_{ML} - \theta_0)^T\right] = J^{-1}\ E\left[\underline{f}(\underline{\theta}_0)\ \underline{f}^*(\underline{\theta}_0)\right] J^{-1}$$

$$= J^{-1} \tag{3.4-13a}$$

or

$$\mathrm{VAR}(\hat{\theta}_i)_{ML} = (J^{-1})_{ii} \ . \tag{3.4-13b}$$

Thus we have shown that the multi-parameter MLE is an efficient, unbiased estimator for large observation time.

3.5 MULTISENSOR, MULTITARGET PARAMETER ESTIMATION

In this section we investigate the detailed structure of the optimum signal processor for multiple parameter estimation. For the purpose of target localization, the parameter set of interest is time delays, range, and bearing. There are two possible approaches in estimating the localization parameters. The first is via a geometric transformation from measured time delays; the second is via a direct range and bearing signal processor. From the results of this study, we will clarify the relative merits between these two approaches. In Section 3.5.1, we derive the optimum multisensor, multitarget time delay processor. In Section 3.5.2 we discuss the various methodologies of obtaining range and bearing. Finally, in Section 3.5.3 we briefly discuss the problem of optimum time delay estimation with unknown target power spectra.

3.5.1 Time Delay Estimation

It was shown in Section 3.3 that the MLE of the time delay parameter vector $\underline{\theta}$ required the simultaneous solution of the vector likelihood equation:

$$\frac{\partial \Lambda(\theta)}{\partial \theta_j} = - \sum_{k=1}^{B} \frac{\partial \Lambda_k(\theta)}{\partial \theta_j} = 0; \qquad j = 1, 2, \ldots, J(M - 1) \tag{3.5.1-1a}$$

where

$$\frac{\partial \Lambda_k(\underline{\theta})}{\partial \theta_j} = \mathrm{tr}\left(\Gamma_k^{-1} \frac{\partial R_k}{\partial \theta_j}\right) + \underline{\alpha}_k^{*} \frac{\partial R_k^{-1}}{\partial \theta_j} \underline{\alpha}_k \; . \tag{3.5.1-1b}$$

We will explore the detailed structure of the optimum signal processor by simplifying Equation (3.5.1-1a). Recall from Equation (3.2-6) that

$$R_k = S_{kj} P_{kj} + \tilde{N}_{kj} \tilde{Q}_{kj} \tag{3.5.1-2a}$$

where

$$\tilde{Q}_{kj} = \left(\sum_{\substack{i=1 \\ i \neq j}}^{J} S_{ki} \, P_{ki} + N_k \, Q_k \right) / \, \tilde{N}_{kj} \tag{3.5.1-2b}$$

and

$$\tilde{N}_{kj} = \sum_{\substack{i=1 \\ i \neq j}}^{J} S_{ki} + N_k \, . \tag{3.5.1-2c}$$

For notational simplicity we shall assume in Equation (3.5.1-1a) that there are J targets but with two sensors. The θ_j's in this case correspond to the time delay for each target. In the case of more than two sensors, θ_j must be replaced by each element of the inter-sensor time delay vector $(\tau_{1j}, \tau_{2j}, \ldots, \tau_{M-1,j})$ of target j, where τ_{ij} denotes the time delay between sensors i + 1 and i of target j.

Applying the well-known matrix inversion Lemma $(H^T R^{-1} H + M^{-1})^{-1} = M - MH^T(HMH^T + R)^{-1} HM$ to R_k^{-1}, we obtain

$$R_k^{-1} = (S_{kj} \, P_{kj} + \tilde{N}_{kj} \, \tilde{Q}_{kj})^{-1}$$

$$= \frac{\tilde{Q}_{kj}^{-1}}{\tilde{N}_{kj}} - \frac{S_{kj}/\tilde{N}_{kj}^2}{1 + G_{kj} \, S_{kj}/\tilde{N}_{kj}} \, \tilde{Q}_{kj}^{-1} \, P_{kj} \, \tilde{Q}_{kj}^{-1} \tag{3.5.1-3}$$

where $G_{kj} = \underline{V}_{kj}^{*} \, \tilde{Q}_{kj}^{-1} \, \underline{V}_{kj}$ is defined as the effective array gain for target j. The derivative of R_k^{-1} w.r.t. θ_j can be written as

$$\frac{\partial R_k^{-1}}{\partial \theta_j} = -|\tilde{h}_{kj}|^2 \, \tilde{Q}_{kj}^{-1} \left(\frac{\partial P_{kj}}{\partial \theta_j} - \tilde{a}_{kj} \, \frac{\partial G_{kj}}{\partial \theta_j} \, P_{kj} \right) Q_{kj}^{-1} \tag{3.5.1-4a}$$

where we have defined

$$|\tilde{h}_{kj}|^2 = \frac{S_{kj}/\tilde{N}_{kj}^2}{1 + G_{kj}\, S_{kj}/\tilde{N}_{kj}} \tag{3.5.1-4b}$$

$$\tilde{a}_{kj} = \tilde{N}_{kj}|\tilde{h}_{kj}|^2 \; . \tag{3.5.1-4c}$$

Using Equations (3.5.1-3) and (3.5.1-4a), the first term in Equation (3.5.1-1b) can be written as follows:

$$\mathrm{tr}\left(R_k^{-1}\,\frac{\partial R_k}{\partial \theta_j}\right) = \frac{S_{kj}}{\tilde{N}_{kj}}\,\mathrm{tr}\left(\tilde{Q}_{kj}^{-1}\,\frac{\partial P_{kj}}{\partial \theta_j}\right) - S_{kj}|\tilde{h}_{kj}|^2\,\mathrm{tr}\left(\tilde{Q}_{kj}^{-1}\,P_{kj}\,\tilde{Q}_{kj}^{-1}\,\frac{\partial P_{kj}}{\partial \theta_j}\right) \; . \tag{3.5.1-5a}$$

But this can be further simplified using the following relations:

$$\begin{aligned}
\mathrm{tr}\left(\tilde{Q}_{kj}^{-1}\,\frac{\partial P_{kj}}{\partial \theta_j}\right) &= \mathrm{tr}\left[\tilde{Q}_{kj}^{-1}\left(\frac{\partial \underline{V}_{kj}}{\partial \theta_j}\,\underline{V}_{kj}^{*} + \underline{V}_{kj}\,\frac{\partial \underline{V}_{kj}^{*}}{\partial \theta_j}\right)\right] \\
&= \underline{V}_{kj}^{*}\,\tilde{Q}_{kj}^{-1}\,\frac{\partial \underline{V}_{kj}}{\partial \theta_j} + \frac{\partial \underline{V}_{kj}^{*}}{\partial \theta_j}\,\tilde{Q}_{kj}^{-1}\,\underline{V}_{kj} \\
&= \frac{\partial G_{kj}}{\partial \theta_j}
\end{aligned} \tag{3.5.1-5b}$$

and

$$\begin{aligned}
\mathrm{tr}\left(\tilde{Q}_{kj}^{-1}\,P_{kj}\,\tilde{Q}_{kj}^{-1}\,\frac{\partial P_{kj}}{\partial \theta_j}\right) &= \mathrm{tr}\left[\tilde{Q}_{kj}^{-1}\,\underline{V}_{kj}\,\underline{V}_{kj}^{*}\,\tilde{Q}_{kj}^{-1}\left(\frac{\partial \underline{V}_{kj}}{\partial \theta_j}\,\underline{V}_{kj}^{*} + \underline{V}_{kj}\,\frac{\partial \underline{V}_{kj}^{*}}{\partial \theta_j}\right)\right] \; . \\
&= G_{kj}\,\frac{\partial G_{kj}}{\partial \theta_j} \; .
\end{aligned} \tag{3.5.1-5c}$$

Thus Equation (3.5.1-5a) reduces to

$$\mathrm{tr}\left(R_k^{-1}\frac{\partial R_k}{\partial\theta_j}\right) = \frac{S_{kj}}{\tilde{N}_{kj}}\frac{\partial G_{kj}}{\partial\theta_j} - S_{kj}\,|\tilde{h}_{kj}|^2\,G_{kj}\frac{\partial G_{kj}}{\partial\theta_j}$$

$$= \tilde{a}_{kj}\frac{\partial G_{kj}}{\partial\theta_j}\,. \tag{3.5.1-6}$$

Using Equations (3.5.1-1b), (3.5.1-4a) and (3.5.1-5c) in (3.5.1-1a), the likelihood equation becomes

$$\frac{\partial\Lambda(\theta)}{\partial\theta_j} = \sum_{k=1}^{B}|\tilde{h}_{kj}|^2\,\underline{\alpha}_k^*\,\tilde{Q}_{kj}^{-1}\left(\frac{\partial P_{kj}}{\partial\theta_j} - \tilde{a}_{kj}\frac{\partial G_{kj}}{\partial\theta j}P_{kj}\right)\tilde{Q}_{kj}^{-1}\,\underline{\alpha}_k - \tilde{a}_{kj}\frac{\partial G_{kj}}{\partial\theta_j}$$

$$= 0. \qquad j = 1, 1 \ldots J \tag{3.5.1-7}$$

Equation (3.5.1-7) reduces to that obtained by Bangs (References 21, 23) for the single target case. It should be pointed out that our development up to here in many ways parallels Bang's work. However, there are also major differences. We are interested in a multitarget environment while Bangs' work dealt exclusively with single target. We are interested in joint time delay vector estimation while Bangs' work is primarily concerned with range and bearing estimation. Thus, our work in this section can be considered as an extension of Bangs' original work to include the multitarget, multisensor environment.

For time delay estimation, the likelihood equation (Equation (3.5.1-7)) can be further simplified as follows.

Recall from Equations (3.2-4e) and (3.2-4j) that the steering vector for target j is

$$\underline{V}_{kj} = \left(1\; e^{j\omega_k\underline{U}_1^T\underline{\tau}_j}\; e^{j\omega_k\underline{U}_2^T\underline{\tau}_j}\;\ldots\; e^{j\omega_k\underline{U}_{M-1}^T\underline{\tau}_j}\right)^T. \tag{3.5.1-8a}$$

where the attenuation coefficients a_{ij} were assumed known and for convenience they were assumed to have unity.

Therefore, the mn element of P_{kj} is

$$P_{kj}^{mn} = e^{j\omega_k(\underline{U}_{m-1}-\underline{U}_{n-1})^T \underline{\tau}_j} \quad , \tag{3.5.1-8b}$$

but

$$\frac{\partial P_{kj}^{mn}}{\partial \tau_{ij}} = j\omega_k(\underline{U}_{m-1} - \underline{U}_{n-1})^T \frac{\partial \underline{\tau}_j}{\partial \tau_{ij}} \left[e^{j\omega_k(U_{m-1}-U_{n-1})^T \underline{\tau}_j}\right]$$

$$= j\omega_k \Phi_i^{mn} \quad P_{kj}^{mn} \tag{3.5.1-8c}$$

where $\Phi^{mn} = (\underline{U}_{m-1} - \underline{U}_{n-1})^T \ \underline{\tau}_j$ and Φ_i^{mn} is defined as the mn element of the matrix Φ_i given by

$$\Phi_i^{mn} = \frac{\partial \Phi^{mn}}{\partial \tau_{ij}}$$

$$= (\underline{U}_{m-1} - \underline{U}_{n-1})^T \frac{\partial \underline{\tau}_j}{\partial \tau_{ij}} \ ; \qquad \begin{array}{l} i = 1, 2, \ldots, M-1 \\ m = 1, 2, \ldots, M \\ n = 1, 2, \ldots, M \end{array}$$

$$= \begin{cases} 1 & ; \text{ if } n \leq i \leq m-1 \\ -1 & ; \text{ if } m \leq i \leq n-1 \\ 0 & ; \text{ otherwise.} \end{cases} \tag{3.5.1-8d}$$

Note that for a system of M sensors, Φ_i is an M x M matrix. For example, let M = 3, then

$$\Phi_1 = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \ ; \qquad \Phi_2 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix} .$$

Define 1_M as an M x M square matrix of ones, and V_{kj} as a diagonal "steering matrix", whose diagonal elements correspond to the elements of the steering vector $\underline{V}_{kj}$; i.e.,

$$V_{kj} = \text{diag}\left\{1,\ e^{j\omega_k \underline{U}_1^T \underline{\tau}_j},\ e^{j\omega_k \underline{U}_2^T \underline{\tau}_j},\ \ldots,\ e^{j\omega_k \underline{U}_{M-1}^T \underline{\tau}_j}\right\}. \tag{3.5.1-9}$$

Then the following relations can be easily established:

$$P_{kj} = V_{kj}\ 1_M\ V_{kj}^* \tag{3.5.1-10a}$$

and

$$\frac{\partial P_{kj}}{\partial \tau_{ij}} = j\omega_k\ V_{kj}\ \Phi_i\ V_{kj}^*\ . \tag{3.5.1-10b}$$

Furthermore, using Equation (3.5.1-10b), Equation (3.5.1-5b) can be written as

$$\frac{\partial G_{kj}}{\partial \tau_{ij}} = j\omega_k\ \text{tr}(\tilde{Q}_{kj}^{-1}\ V_{kj}\ \Phi_i\ V_{kj}^*)\ . \tag{3.5.1-10c}$$

Finally, using Equations (3.5.1-10a), (3.5.1-10b), (3.5.1-10c) and replacing θ_j by τ_{ij}, the likelihood equation (Equation (3.5.1-7)) can be simplified to

$$\frac{\partial \Lambda(\underline{\tau})}{\partial \tau_{ij}} = \sum_{k=1}^{B} j\omega_k \left[|\tilde{h}_{kj}|^2\ \underline{\alpha}_k^*\ \tilde{Q}_{kj}^{-1}\ V_{kj}\left(\Phi_i - \tilde{b}_i^{kj}\ 1_M\right) V_{kj}^*\ \tilde{Q}_{kj}^{-1}\ \underline{\alpha}_k - \tilde{b}_i^{kj} \right]$$

$$= 0 \tag{3.5.1-11a}$$

where

$$\tilde{b}_i^{kj} = \tilde{a}_{kj}\ \text{tr}(\tilde{Q}_{kj}^{-1}\ V_{kj}\ \Phi_i\ V_{kj}^*) \tag{3.5.1-11b}$$

is the bias correction term and for $i = 1, 2, \ldots, M-1$ and $j = 1, 2, \ldots, J$ where M is the number of sensors and J is the number of targets. Note that the J(M - 1) equations are coupled and must be solved jointly for the stationary point. For long observation time such that the frequency samples are dense over the frequency bands of the signal and the noise, the summation in Equation (3.5.1-11a) can be replaced by integration as follows:

$$Z_{ij}(\underline{\tau}) = \frac{\partial \Lambda(\underline{\tau})}{\partial \tau_{ij}}$$

$$= \frac{T}{2\pi} \int_0^{\infty} j\omega \left\{ |\tilde{h}_j(\omega)|^2 \, \underline{\alpha}^*(\omega) \, \tilde{Q}_j^{-1}(\omega) \, V_j(\omega) \left[\Phi_i - b_i^j(\omega) \, 1_M\right] \right.$$

$$\left. V_j^*(\omega) \, \tilde{Q}_j^{-1}(\omega) \, \underline{\alpha}(\omega)/T - \tilde{b}_i^j(\omega) \right\} d\omega \; ; \quad \begin{array}{l} i = 1, 2, \ldots, M-1 \\ j = 1, 2, \ldots, J. \end{array}$$

$$= 0 \tag{3.5.1-12a}$$

Note that in obtaining Equation (3.5.1-12a) the relations $T\alpha_k = \alpha(\omega)$ and $|\tilde{h}_k|^2 = T|\tilde{h}(\omega)|^2$ have been used.

It is recalled that for the jth target

$$|\tilde{h}_j(\omega)|^2 = \frac{S_j(\omega)/\tilde{N}_j^2(\omega)}{1 + G_j(\omega) \, S_j(\omega)/\tilde{N}_j(\omega)} \tag{3.5.1-12b}$$

$$\tilde{a}_j(\omega) = \tilde{N}_j(\omega) \, |\tilde{h}_j(\omega)|^2 \tag{3.5.1-12c}$$

$$\tilde{b}_i^j(\omega) = \tilde{a}_j(\omega) \, \mathrm{tr}[\tilde{Q}_j^{-1}(\omega) \, V_j(\omega) \, \Phi_i(\omega) \, V_j^*(\omega)] \tag{3.5.1-12d}$$

$$\tilde{Q}_j^{-1}(\omega) = \tilde{N}_j(\omega)\left[\sum_{\substack{i=1\\ i\neq j}}^{J} S_i(\omega)\; P_i(\omega) + N(\omega)\; Q(\omega)\right]^{-1} \tag{3.5.1-12e}$$

$$\tilde{N}_j(\omega) = \sum_{\substack{i=1\\ i\neq j}}^{J} S_i(\omega) + N(\omega) \tag{3.5.1-12f}$$

$$P_j(\omega) = V_j(\omega)\; 1_M\; V_j^*(\omega) \tag{3.5.1-12g}$$

and

$$G_j(\omega) = \underline{V}_j^*(\omega)\; \tilde{Q}_j^{-1}(\omega)\; V_j(\omega)\ . \tag{3.5.1-12h}$$

The optimum multitarget, multisensor time delay signal processor is shown in Figure 3-1. There is a total of $J(M - 1)$ processing channels for the case of M sensors and J targets. For simplicity we show a single processing channel. Note that the processing channels are tightly coupled. The signal conditioning filters depend on the time delay parameters from other processing channels as well. This is an order of magnitude more complex compared to a single target case. A number of suboptimal realizations can be found as discussed in Section 4. Finally, we remark that for convenience, we show the optimum processor (Figure 3-1) in the continuous frequency domain. For practical considerations, the discrete counterpart, Equation (3.5.1-11a), is normally used since the correlation process can be mechanized easily via the Fast Fourier Transform (FFT) algorithm.

Using Equations (3.5.1-12b-f), we shall study the specific structure of the optimum time delay processor for a number of simple but important cases.

3.5.1.1 <u>Case 1: One Target and Two Sensors (J = 1, M = 2)</u>. For convenience we assume $Q(\omega) = I$; i.e., the noise processes are equal in power and

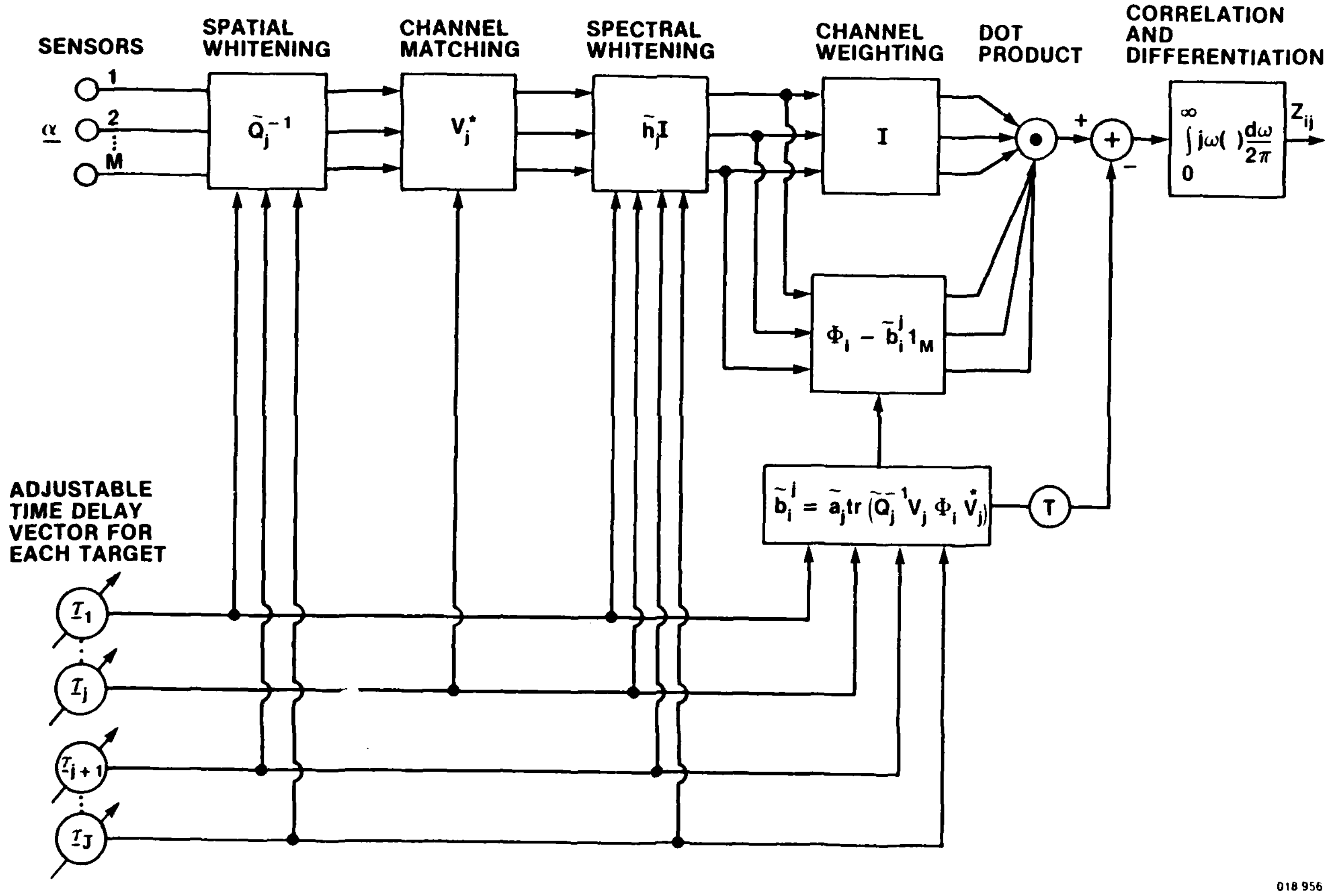

Figure 3-1. A Multisensor, Multitarget Time Delay Processing Channel

uncorrelated between sensors. Noise processes being uncorrelated between sensors is a reasonable assumption since in practice sensors are separated at least at a half wavelength spacing.

The steering vector is $\underline{v} = (1 \ e^{j\omega\tau})^T$. The following relations can be verified easily:

$$\tilde{Q}^{-1}(\omega) = Q^{-1}(\omega) = I \tag{3.5.1-13a}$$

$$G(\omega) = \underline{v}^* \ Q^{-1} \ \underline{v} = 2 \tag{3.5.1-13b}$$

$$b(\omega) = tr(Q^{-1} \ V \ \Phi_1 \ V^*) = 0 \tag{3.5.1-13c}$$

$$|\tilde{h}(\omega)|^2 = \frac{S(\omega)/N^2(\omega)}{1 + 2 \ S(\omega)/N(\omega)} \tag{3.5.1-13d}$$

and

$$\Phi_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} ; \ V = \begin{bmatrix} 1 & 0 \\ 0 & e^{j\omega\tau} \end{bmatrix} . \tag{3.5.1-13e}$$

Thus from Equation (3.5.1-12a), the likelihood equation is

$$Z(\tau) = \frac{1}{2\pi} \int_0^\infty j\omega|\tilde{h}|^2 \ \underline{\alpha}^* \ V \ \Phi_1 \ V^* \ \underline{\alpha} \ d\omega = 0 \ . \tag{3.5.1-14a}$$

This simple time delay processor is diagrammed in Figure 3-2. This processor is identical to the one studied by Carter (Reference 22) and can be shown as follows. From Equation (3.5.1-14a), we have

$$Z(\tau) = \frac{1}{2\pi} \int_0^\infty j\omega|\tilde{h}|^2 \ \underline{\alpha}^* \begin{bmatrix} 1 & 0 \\ 0 & e^{j\omega\tau} \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & e^{-j\omega\tau} \end{bmatrix} \underline{\alpha} \ d\omega$$

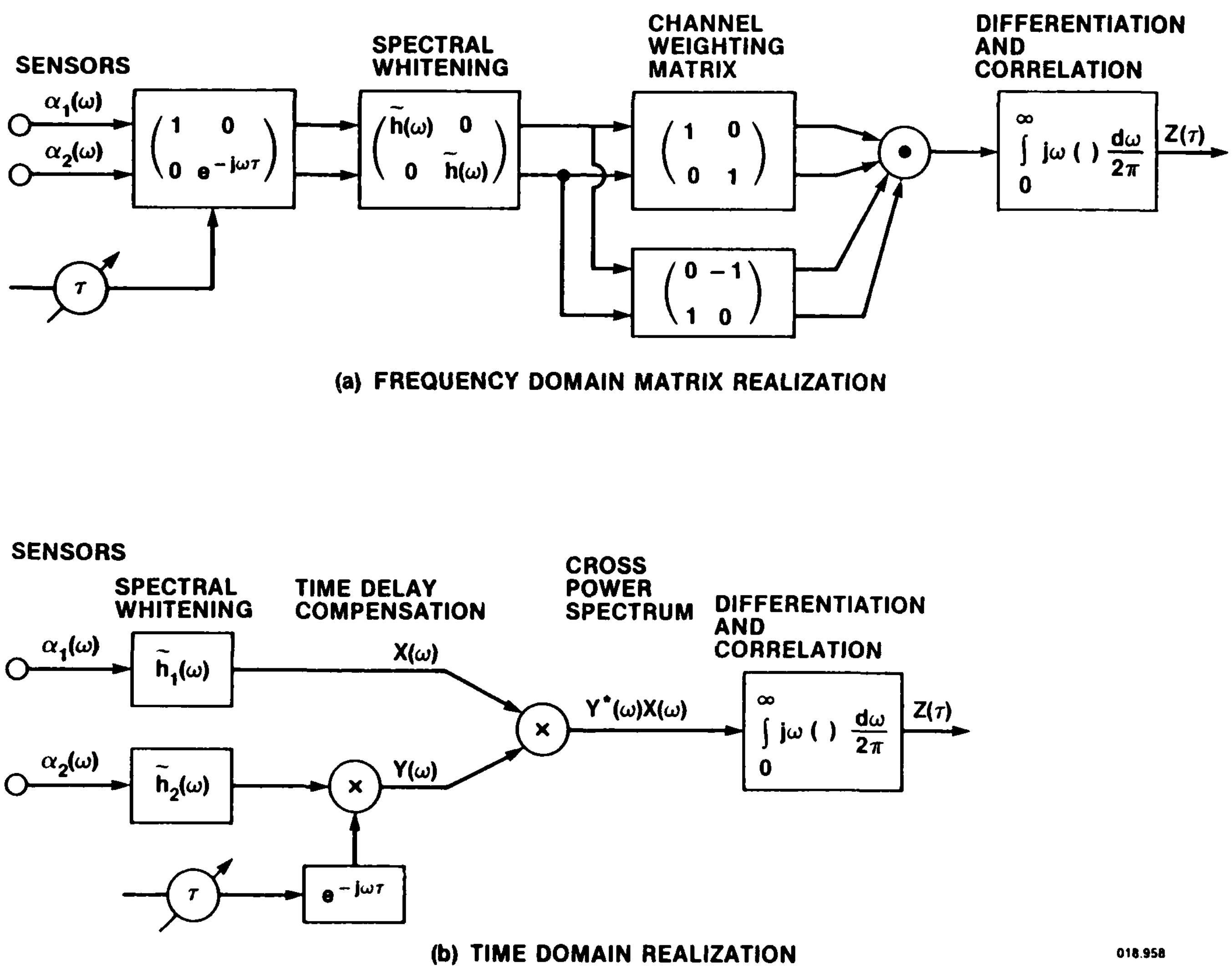

Figure 3-2. Optimum Two-Sensor, One-Target Time Delay Processor

$$= \frac{T}{2\pi} \int_{-\infty}^{\infty} j\omega |\tilde{h}|^2 \frac{\alpha_1(\omega)\ \alpha_2^*(\omega)}{T}\ e^{j\omega\tau}\, d\omega$$

$$= \frac{T}{2\pi} \int_{-\infty}^{\infty} j\omega |\tilde{h}|^2\ G_{12}(\omega)\ e^{j\omega\tau}\, d\omega$$

$$= T \frac{\partial}{\partial\tau} \int_{-\infty}^{\infty} |\tilde{h}|^2\ G_{12}\ (\omega)\ e^{j\omega\tau} \frac{d\omega}{2\pi}$$

$$= T \frac{\partial}{\partial\tau} R_{12}\ (\tau) \tag{3.5.1-14b}$$

where G_{12} is the estimated cross power spectrum between channels $\alpha_1(\omega)$ and $\alpha_2(\omega)$, and $R_{12}(\tau)$ is the GCC function studied by Knapp and Carter (Reference 1). A block diagram of this processor is shown in Figure 3-2(b). Note that the null of $Z(\tau)$ corresponds to the peak of $R_{12}(\tau)$. Furthermore, the discrete form of Equation (3.5.1-14b) is

$$Z(\tau) = \sum_{k=-B}^{B} j\omega_k |h_k|^2\ \alpha_{1k}\ \alpha_{2k}^*\ e^{j\omega_k\tau} \tag{3.5.1-14c}$$

$$= \frac{\partial}{\partial\tau} \sum_{k=-B}^{B} |h_k|^2\ \alpha_{1k}\ \alpha_{2k}^*\ e^{j\omega_k\tau}\ . \tag{3.5.1-14d}$$

From Equation (3.5.1-14b), we note that the GCC function is directly proportional to the likelihood function. The optimum estimate is determined by locating the peak of the GCC function or, equivalently, the null of its derivative. We have shown in Section 3.4 that the MLE is efficient for a long observation time. We have also obtained a general closed form expression for the CRLB. The CRLB for this case can be determined easily. From Equations (3.4-1) and (3.4-5) we have

$$\mathrm{VAR}(\hat{\tau}) \geq \left[\sum_{k=1}^{B} \mathrm{tr}\left(- \frac{\partial R_k^{-1}}{\partial \tau} \frac{\partial R_k}{\partial \tau} \right) \right]^{-1} . \tag{3.5.1-15a}$$

Now combining Equations (3.5.1-4a) and (3.5.1-10b), we obtain

$$\frac{\partial R_k}{\partial \tau} = S_k \frac{\partial P_k}{\partial \tau} = j\omega_k \; S_k \; V_k \; \Phi_1 \; V_k^* \tag{3.5.1-15b}$$

$$\frac{\partial R_k^{-1}}{\partial \tau} = -j\omega_k |\tilde{h}_k|^2 \; V_k \; \Phi_1 \; V_k^* \; . \tag{3.5.1-15c}$$

Thus

$$\begin{aligned} \mathrm{tr}\left(- \frac{\partial R_k^{-1}}{\partial \tau} \frac{\partial R_k}{\partial \tau} \right) &= -\omega_k^2 |\tilde{h}_k|^2 \; S_k \; \mathrm{tr}(V_k \; \Phi_1^2 \; V_k^*) \\ &= 2\omega_k^2 |\tilde{h}_k|^2 \; S_k \end{aligned} \tag{3.5.1-15d}$$

and the CRLB is

$$\mathrm{VAR}(\hat{\tau}) \geq \left(2 \sum_{k=1}^{B} \omega_k^2 \frac{S_k^2/N_k^2}{1 + 2\; S_k/N_k} \right)^{-1} \tag{3.5.1-16a}$$

$$\geq 2\pi \left(2T \int_0^{\infty} \frac{c(\omega)}{1 - c(\omega)} \, \omega^2 d\omega \right)^{-1} \tag{3.5.1-16b}$$

which is identical to the expression derived by Carter (Reference 2). Note that T is the observation time and

$$c(\omega) = \left(\frac{S(\omega)}{S(\omega) + N(\omega)} \right)^2 \tag{3.5.1-16c}$$

is known as the magnitude square coherence (MSC) function for the case of two equal-noise power channels.

3.5.1.2 Case 2: Two Targets and Two Sensors ($J = 2$, $M = 2$). Again assuming $Q_k = I$ for simplicity. The parameter vector consists of two elements; i.e., $\underline{\theta} = (\tau_1, \tau_2)^T$, the time delays to target number one and target number two. The optimum estimates can be obtained by solving simultaneously the two likelihood equations:

$$\frac{\partial\Lambda(\tau_1, \tau_2)}{\partial\tau_1} = \int_0^\infty j\omega\{|\tilde{h}_1|^2 \ \underline{\alpha}^* \ \tilde{Q}_1^{-1} \ V_1 \ [\Phi_1 - \tilde{b}_1 \ 1_M] \ V_1^* \ \tilde{Q}_1^{-1} \ \underline{\alpha} - T \ \tilde{b}_1\}\frac{d\omega}{2\pi} = 0 \tag{3.5.1-17a}$$

$$\frac{\partial\Lambda(\tau_1, \tau_2)}{\partial\tau_2} = \int_0^\infty j\omega\{|\tilde{h}_2|^2 \ \underline{\alpha}^* \ \tilde{Q}_2^{-1} \ V_2 \ [\Phi_1 - \tilde{b}_2 \ 1_M] \ V_2^* \ \tilde{Q}_2^{-1} \ \underline{\alpha} - T \ \tilde{b}_2\}\frac{d\omega}{2\pi} = 0 \tag{3.5.1-17b}$$

where for notational simplicity, we have suppressed the frequency dependency. The two likelihood equations can be further simplified using the following relations:

$$|\tilde{h}_1|^2 = \frac{S_1/(S_2 + N)^2}{1 + G_1 \ S_1/(S_2 + N)} \tag{3.5.1-17c}$$

$$\tilde{a}_1 = \frac{S_1/(S_2 + N)}{1 + G_1 \ S_1/(S_2 + N)} \tag{3.5.1-17d}$$

$$\tilde{b}_1 = \tilde{a}_1 \ \mathrm{tr}\left(\tilde{Q}_1^{-1} \ \frac{\partial P_1}{\partial\tau_1}\right) = -\tilde{a}_1 \ \tilde{c}_2 \ \mathrm{tr}[(V_2^* \ V_1)^* \ 1_M \ (V_2^* \ V_1)\Phi_1] \tag{3.5.1-17e}$$

$$\tilde{c}_2 = \frac{(1 + S_2/N)\ S_2/N}{1 + 2\ S_2/N} \tag{3.5.1-17f}$$

$$G_1 = 2(1 + S_2/N) - \tilde{c}_2\ |\underline{v}_1^*\ \underline{v}_2|^2 \tag{3.5.1-17g}$$

and

$$\Phi_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} . \tag{3.5.1-17h}$$

By exchanging indices between 1 and 2, we obtain a similar set of equations for the second processor channel. Therefore, a simpler form of the likelihood equation is:

$$\frac{\partial\Lambda(\tau_1,\ \tau_2)}{\partial\tau_1} = \frac{T}{2\pi}\int_0^{\infty} j\omega\ e^{j\omega\tau_1}\ d\omega \left[|\tilde{h}_1|^2\ \tilde{G}_{x_1x_2}(\omega) + \tilde{a}_1\ \tilde{c}_2(1 + \frac{1}{T}|\tilde{h}_1\ \underline{v}_1^*\ \tilde{Q}_1^{-1}\ \underline{\alpha}|^2)e^{-j\omega\tau_2}\right] = 0 \tag{3.5.1-18a}$$

$$\frac{\partial\Lambda(\tau_1,\ \tau_2)}{\partial\tau_2} = \frac{T}{2\pi}\int_0^{\infty} j\omega\ e^{j\omega\tau_2}\ d\omega \left[|\tilde{h}_2|^2\ \tilde{G}_{y_1y_2}(\omega) + \bar{a}_2\ c_1(1 + \frac{1}{T}|\tilde{h}_2\ \underline{v}_2^*\ \tilde{Q}_2^{-1}\ \underline{\alpha}|^2)\ e^{-j\omega\tau_1}\right] = 0 \tag{3.5.1-18b}$$

where

$$\underline{X} = \tilde{Q}_1^{-1}\ \underline{\alpha}$$

$$Y = \tilde{Q}_2^{-1}\ \underline{\alpha}$$

and

$$\tilde{G}_{x_1x_2}(\omega) = \frac{X_1(\omega)\ X_2^*(\omega)}{T}$$

$$\tilde{G}_{y_1y_2}(\omega) = \frac{Y_1(\omega)\ Y_2^*(\omega)}{T}$$

are the cross power spectra, respectively, for $\underline{X}$ and $\underline{Y}$.

Substituting Equations (3.5.1-17g and f) in Equation (3.5.1-17c) yields the optimum two-target, two-sensor spectral shaping filters:

$$|\tilde{h}_1|^2 = \frac{S_1/(S_2+N)^2}{1 + \left(2 - \frac{S_2/N}{1 + 2\ S_2/N}\ |\underline{V}_1^*\ \underline{V}_2|^2\right) S_1/N} \qquad (3.5.1\text{-}18c)$$

and similarly,

$$|\tilde{h}_2|^2 = \frac{S_2/(S_1+N)^2}{1 + \left(2 - \frac{S_1/N}{1 + 2\ S_1/N}\ |\underline{V}_2^*\ \underline{V}_1|^2\right) S_2/N} \ . \qquad (3.5.1\text{-}18d)$$

Note that the filters $|\tilde{h}_1|^2$ and $|\tilde{h}_2|^2$ are a function of the steering vectors, and that the target of interest is treated as part of the signal process and remaining targets are treated as part of the noise process.

A block diagram of this dual channel processor is shown in Figure 3-3. It is seen that the dual processor is tightly coupled. In Chapters 4 and 5 we discuss a number of suboptimum procedures for simplifying the complex structure of this processor in light of improved multitarget time delay resolution.

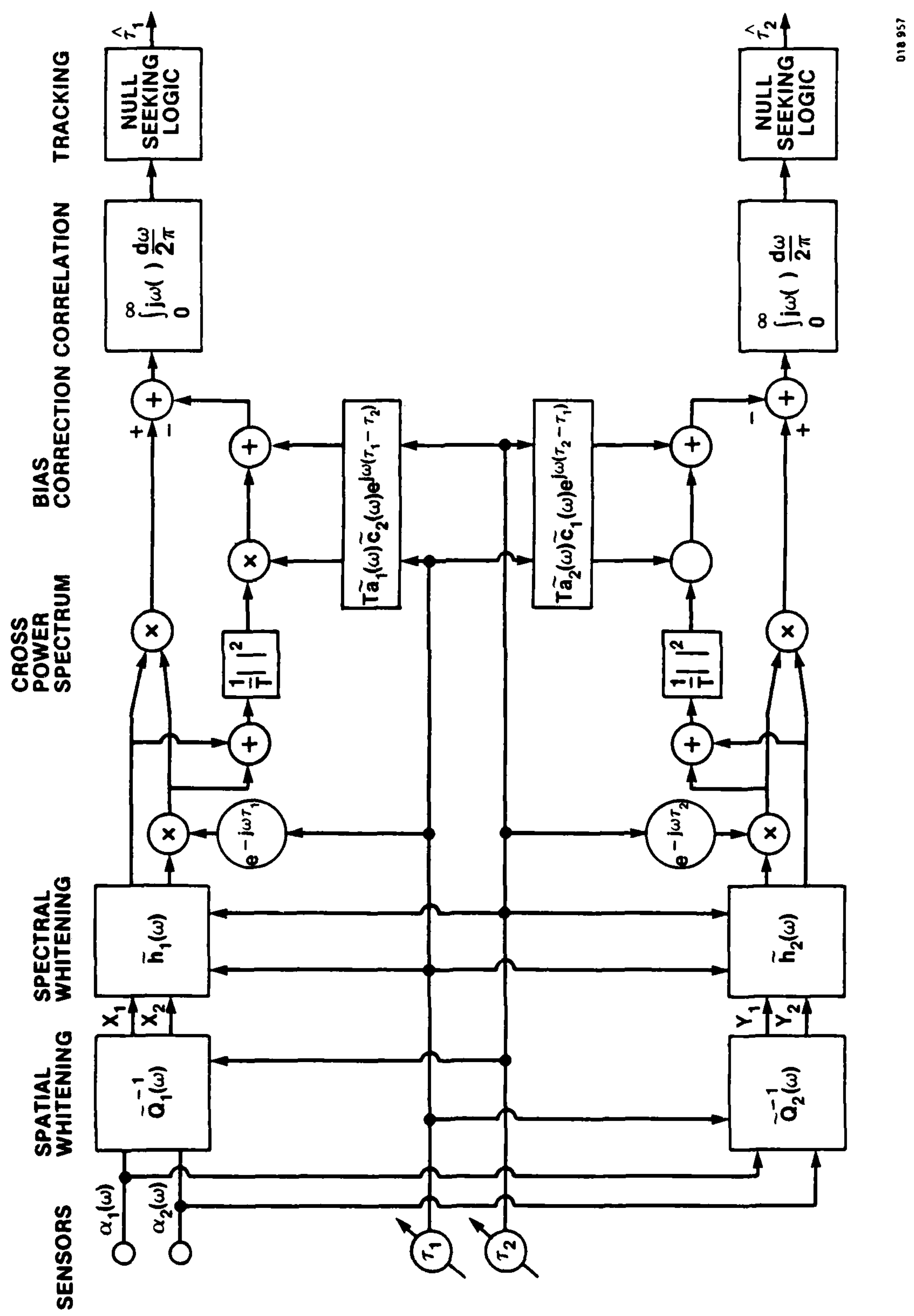

Figure 3-3. Optimum Two-Sensor, Two-Target Time Delay Processor

Using Equations (3.4-1) and (3.4-5), the CRLB of the time delay estimates for the two-target, two-sensor case can be found as follows:

$$\mathrm{VAR}(\hat{\tau}_i) \geq \frac{1}{(1 - M_{12}^2)} \frac{1}{J_{ii}} ; \qquad i = 1, 2 \tag{3.5.1-19a}$$

where M_{12} is the coefficient of mutual dependence given by

$$M_{12} = \frac{J_{12}}{(J_{11}\ J_{22})^{1/2}} \tag{3.5.1-19b}$$

and the quantities J_{ij} are defined by (Equation (3.4-5))

$$J_{ij} = \sum_{k=1}^{B} \left[\mathrm{tr} \left(- \frac{\partial R_k^{-1}}{\partial \tau_i} \frac{\partial R_k}{\partial \tau_j} \right) \right] . \tag{3.5.1-19c}$$

From the relations

$$R_k = S_{k1}\ P_{k1} + S_{k2}\ P_{k2} + N_k\ I \tag{3.5.1-19d}$$

and

$$- \frac{\partial R_k^{-1}}{\partial \tau_i} = |\tilde{h}_{ki}|^2\ \tilde{Q}_{ki}^{-1} \left(\frac{\partial P_{ki}}{\partial \tau_i} - \tilde{\alpha}_{ki} \frac{\partial G_{ki}}{\partial \tau_i} P_{ki} \right) \tilde{Q}_{ki}^{-1} ; \qquad i = 1, 2 \tag{3.5.1-19e}$$

one obtains after some straightforward but tedious algebraic manipulations: (See Appendix B)

$$J_{11} = \frac{T}{\pi} \int_0^{\infty} \omega^2 \left[\frac{(S_1/N)^2}{1 + G_1 \; S_1/(S_2 + N)} \right] \left[\frac{1 + S_2/N}{1 + 2 \; S_2/N} \right]^2 \gamma_1 \, d\omega \tag{3.5.1-20a}$$

$$J_{12} = \frac{T}{\pi} \int_0^{\infty} \omega^2 \left[\frac{S_1 S_2/N^2}{1 + G_1 \; S_1/(S_2 + N)} \right] \left[\frac{1}{1 + 2 \; S_2/N} \right] \gamma_{12} \, d\omega \tag{3.5.1-20b}$$

$$J_{22} = \frac{T}{\pi} \int_0^{\infty} \omega^2 \left[\frac{(S_2/N)^2}{1 + G_2 \; S_2/(S_1 + N)} \right] \left[\frac{1 + S_1/N}{1 + 2 \; S_1/N} \right]^2 \gamma_2 \, d\omega \; . \tag{3.5.1-20c}$$

Therefore, using Equation (3.5.1-19) the CRLB of the optimum two-sensor two-target time delay estimates is given by

$$\mathrm{VAR}(\hat{\tau}_1) \geq \frac{2\pi}{(1 - M_{12}^2)} \left\{ 2T \int_0^{\infty} \omega^2 \left[\frac{(S_1/N)^2}{1 + G_1 \; S_1/(S_2 + N)} \right] \left[\frac{1 + S_2/N}{1 + 2 \; S_2/N} \right]^2 \gamma_1 \, d\omega \right\}^{-1} \tag{3.5.1-21a}$$

$$\mathrm{VAR}(\hat{\tau}_2) \geq \frac{2\pi}{(1 - M_{12}^2)} \left\{ 2T \int_0^{\infty} \omega^2 \left[\frac{(S_2/N)^2}{1 + G_2 \; S_2/(S_1 + N)} \right] \left[\frac{1 + S_1/N}{1 + 2 \; S_1/N} \right]^2 \gamma_2 \, d\omega \right\}^{-1} \tag{3.5.1-21b}$$

where

$$G_1 = (1 + S_2/N)\left[2 - \left(\frac{S_2/N}{1 + 2\ S_2/N}\right)|\underline{v}_1^* \ \underline{v}_2|^2\right] \tag{3.5.1-22a}$$

$$G_2 = (1 + S_1/N)\left[2 - \left(\frac{S_1/N}{1 + 2\ S_1/N}\right)|\underline{v}_2^* \ \underline{v}_1|^2\right] \tag{3.5.1-22b}$$

and the quantities γ_1, γ_{12} and γ_2 are given by (Appendix B):

$$\gamma_1 = 1 - \left(\frac{S_2/N}{1 + S_2/N}\right)^2 \sum_{n=0}^{3} A_n \cos^n \omega\Delta_{12} \tag{3.5.1-23a}$$

$$\gamma_{12} = \cos \omega\Delta_{12} - 2\left\{\frac{S_1\ S_2/N^2}{[1 + G_1\ S_1/(S_2 + N)]\ (1 + 2\ S_2/N)}\right\}\sin^2 \omega\Delta_{12} \tag{3.5.1-23b}$$

and

$$\gamma_2 = 1 - \left(\frac{S_1/N}{1 + S_1/N}\right)^2 \sum_{n=0}^{3} B_n \cos^n \omega\Delta_{12} \tag{3.5.1-23c}$$

where $\Delta_{12} = \tau_1 - \tau_2$ is the time delay separation between targets 1 and 2. A_n, B_n are defined by the following:

$$A_0 = 4\left[\frac{S_1/N}{1 + G_1\ S_1/(S_2 + N)}\right]\left[\frac{1 + S_2/N}{1 + 2\ S_2/N}\right] - 1 \tag{3.5.1-24a}$$

$$A_1 = -4 \left[\frac{S_1/N}{1 + G_1 \; S_1/(S_2 + N)} \right] \left[\frac{S_2/N}{1 + 2 \; S_2/N} \right] \tag{3.5.1-24b}$$

$$A_2 = 1 - A_0 \tag{3.5.1-24c}$$

$$A_3 = -A_1 \tag{3.5.1-24d}$$

$$B_0 = 4 \left[\frac{S_2/N}{1 + G_2 \; S_2/(S_1 + N)} \right] \left[\frac{1 + S_1/N}{1 + 2 \; S_1/N} \right] - 1 \tag{3.5.1-24e}$$

$$B_1 = -4 \left[\frac{S_1/N}{1 + G_2 \; S_2/(S_1 + N)} \right] \left[\frac{S_2/N}{1 + 2 \; S_1/N} \right] \tag{3.5.1-24f}$$

$$B_2 = 1 - B_0 \tag{3.5.1-24g}$$

$$B_3 = -B_1 \; . \tag{3.5.1-24h}$$

Note that if both targets have identical power spectra ($S_1 = S_2$) and identical time delays ($\tau_1 = \tau_2$) then it can be shown that

$$\gamma_1 = \frac{1 + 2 \; (S_2/N)}{(1 + S_2/N)^2} \tag{3.5.1-25a}$$

$$\gamma_2 = \frac{1 + 2 \; S_1/N}{(1 + S_1/N)^2} \tag{3.5.1-25b}$$

$$\gamma_{12} = 1 \; . \tag{3.5.1-25c}$$

Therefore, using Equations (3.5.1-25a-c) in (3.5.1-20a-c), we obtain $J_{11} = J_{22} = J_{12}$. Hence, $M_{12}^2 = J_{12}^2/(J_{11}J_{22}) = 1$. Thus, the time delay variance is infinite indicating the inherent inappropriateness of a two-target formulation to a one-target problem.

Another observation is that letting the interference power be zero (i.e., $S_2 = 0$), Equation (3.5.1-20a) reduces to Equation (3.5.1-16a), the time delay CRLB for the single target case. Note that because of the following inequality:

$$\int_0^\infty \omega^2 \left[\frac{(S_1/N)^2}{1 + G_1\ S_1/(S_2+N)}\right]\left[\frac{1 + S_2/N}{1 + 2\ S_2/N}\right]^2 \gamma_1\ d\omega$$

$$\leq \frac{1}{(1 - M_{12}^2)} \int_0^\infty \omega^2 \frac{(S_1/N)^2}{1 + 2\ S_1/N}\ d\omega\ , \tag{3.5.1-26}$$

we conclude that the time delay CRLB of the two-target case (Equation (3.5.1-21a)) is always larger than the time delay CRLB of a single target case (Equation (3.5.1-16)). For convenience, let $S = S_1$ be the target power spectrum and $I = S_2$ be the interference power spectrum, then one can define a degradation ratio of a two-target CRLB over a single target CRLB as

$$R_d = \sqrt{\frac{\text{two-target CRLB}}{\text{one-target CRLB}}}$$

$$= \frac{1}{(1 - M_{12}^2)^{1/2}} \left[\frac{\displaystyle\int_0^\infty \omega^2 \frac{(S/N)^2}{1 + 2\ S/N}\ d\omega}{\displaystyle\int_0^\infty \omega^2 \left[\frac{(S/N)^2}{1 + G_1\ S/(I + N)}\right]\left[\frac{1 + I/N}{1 + 2\ I/N}\right]^2 \gamma_1\ d\omega}\right]^{1/2} \tag{3.5.1-27}$$

Note that $R_d \geq 1$ and that R_d is a function of SNR, interference-to-noise ratio (INR), and relative time delay separation.

For the purpose of illustrating the two-sensor, two-target CRLB performance behavior, we assume a one-sided power spectrum of 100 Hz bandwidth centered at 50 Hz. Furthermore, we assume that signal, interference, and noise processes have identical bandwidths. Figures 3-4 and 3-5 present the multitarget shaping function γ_1, and γ_{12}, respectively, as a function of frequency for a number of interference-to-signal ratios (ISRs). Two effects can be seen: (1) the function $\gamma_1(\omega)$ reduces the low frequency band contribution to the CRLB while the function $\gamma_{12}(\omega)$ reduces the high frequency band contribution to the coefficient of mutual dependence; (2) as ISR decreases, both $\gamma_1(\omega)$ and $\gamma_{12}(\omega)$ approach unity. Figure 3-6 presents the two-sensor, two-target degradation ratio (Equation (3.5.1-27)) as a function of time delay separation for a number of ISR values. Note that identical signal, interference, and noise power spectra of 100 Hz bandwidth are used. As shown in Figure 3-6, the degradation ratio is oscillatory as a function of time delay separation, which is plotted in terms of the noise inverse bandwidth. The oscillatory behavior decreases as separation increases. Two major peaks are observed. The first occurs when time delay separation approaches zero and the other occurs at 4 times the inverse bandwidth. Note that the first peak goes to infinity at zero while the second decreases as ISR decreases. Thus, targets with identical spectrum and spatial location yield the worst estimate since they are not separable in frequency and/or space. In order to see the effect of different signal and interference spectra on the resulting estimate, we fix the SIR but vary the interference spectrum. We choose a noise bandwidth of 5000 Hz (B = 5000). The results are presented in Figures 3-7 and 3-8. Two observations can be made: (1) degradation ratio is no longer infinite at zero time delay separation with non-identical signal and interference spectra, and (2) comparison between Figures 3-6 and 3-7 shows that the degradation ratio as a function of time delay separation is bandwidth independent.

3.5.1.3 <u>Case 3: One Target and M Sensors (J = 1, M = M)</u>. For M sensors and with $Q_k = I$, the parameter set consists of M-1 independent time delays.

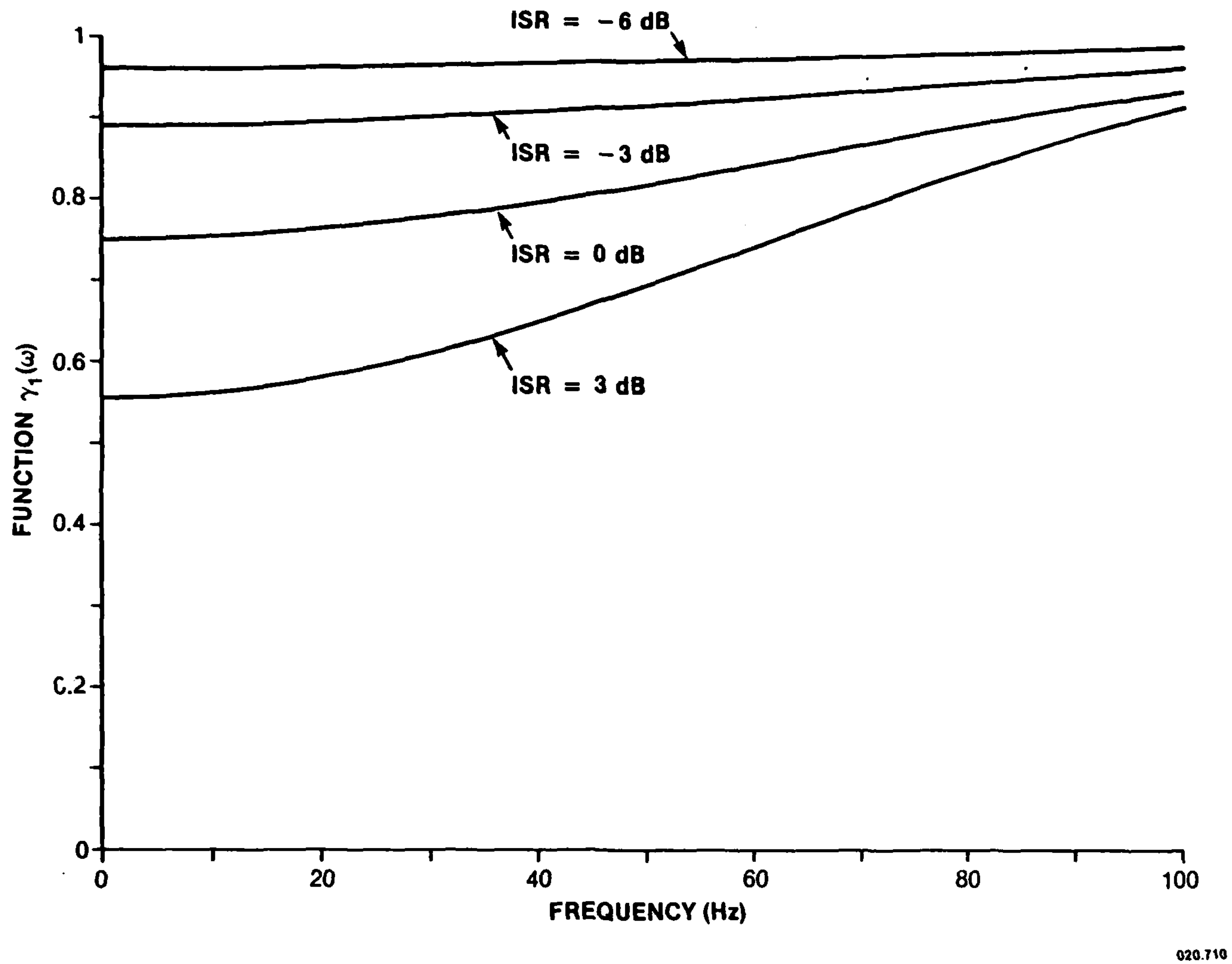

Figure 3-4. Frequency Response of the Function $\gamma_1(\omega)$ at Different Interference-to-Signal Ratios (SNR = 0 dB)

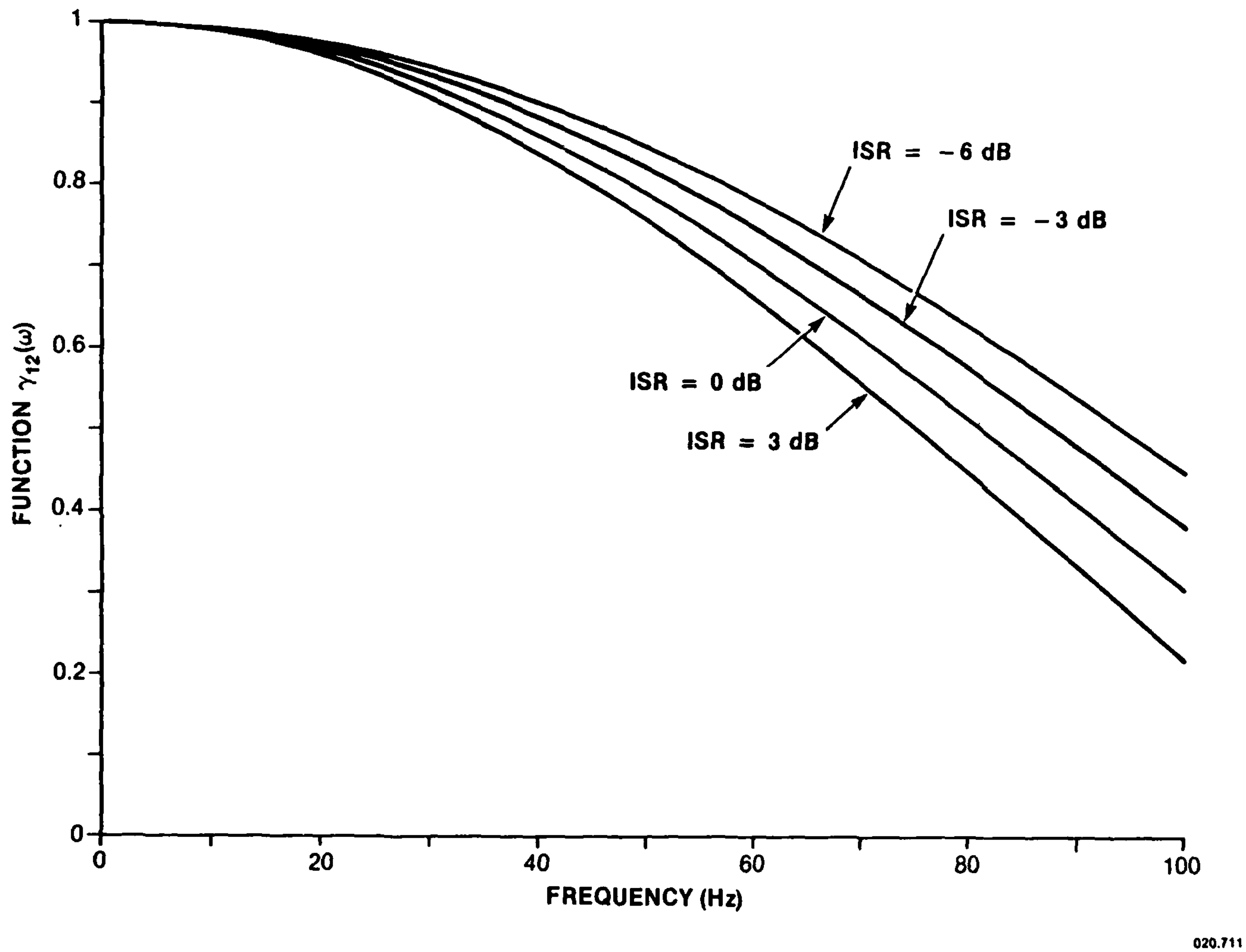

Figure 3-5. Frequency Response of the Function $\gamma_{12}(\omega)$ at Different Interference-to-Signal Ratios (SNR = 0 dB)

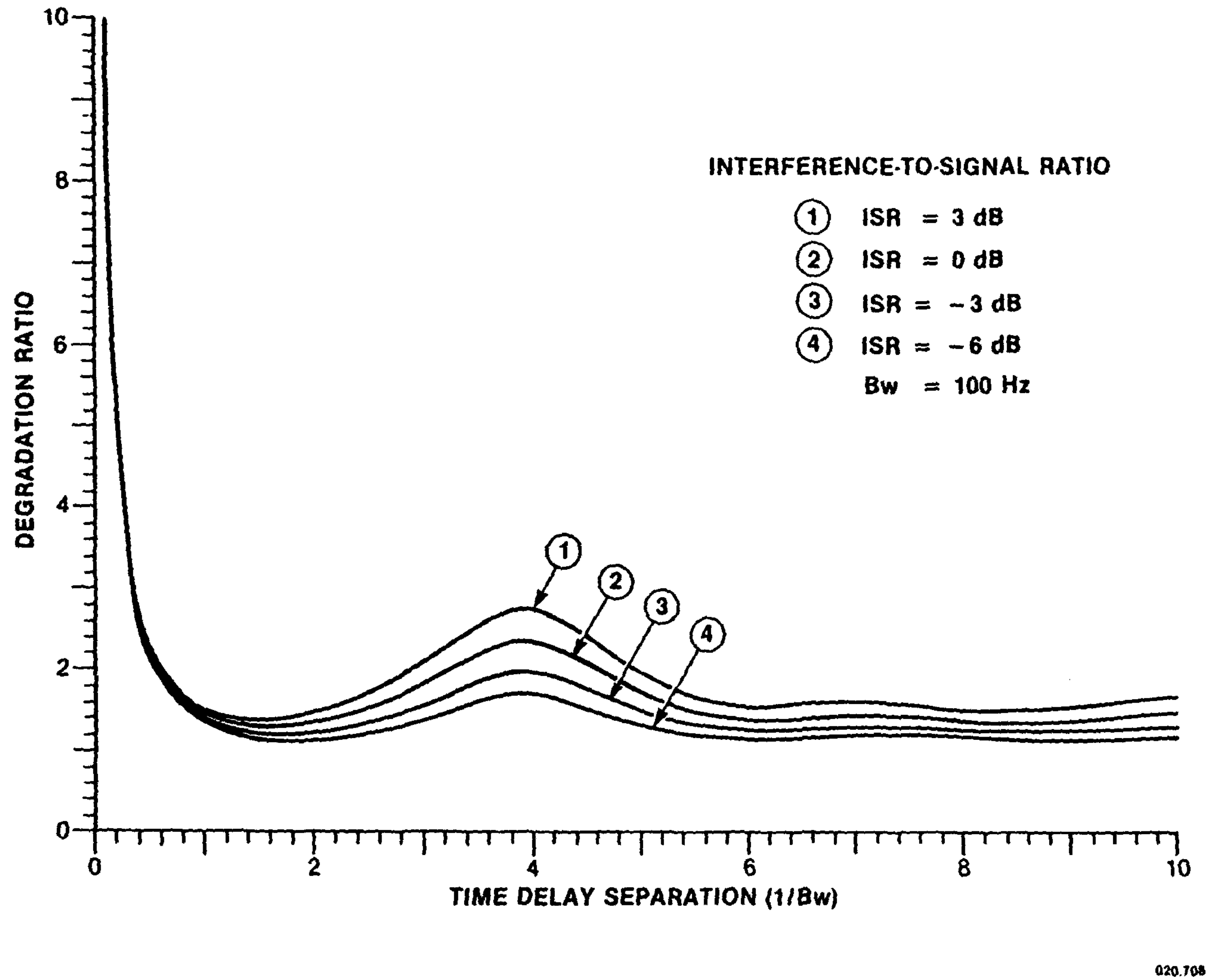

Figure 3-6. Two-Sensor, Two-Target Cramer-Rao Lower Bound Degradation Ratio Versus Time Delay Separation (SNR = 0 dB)

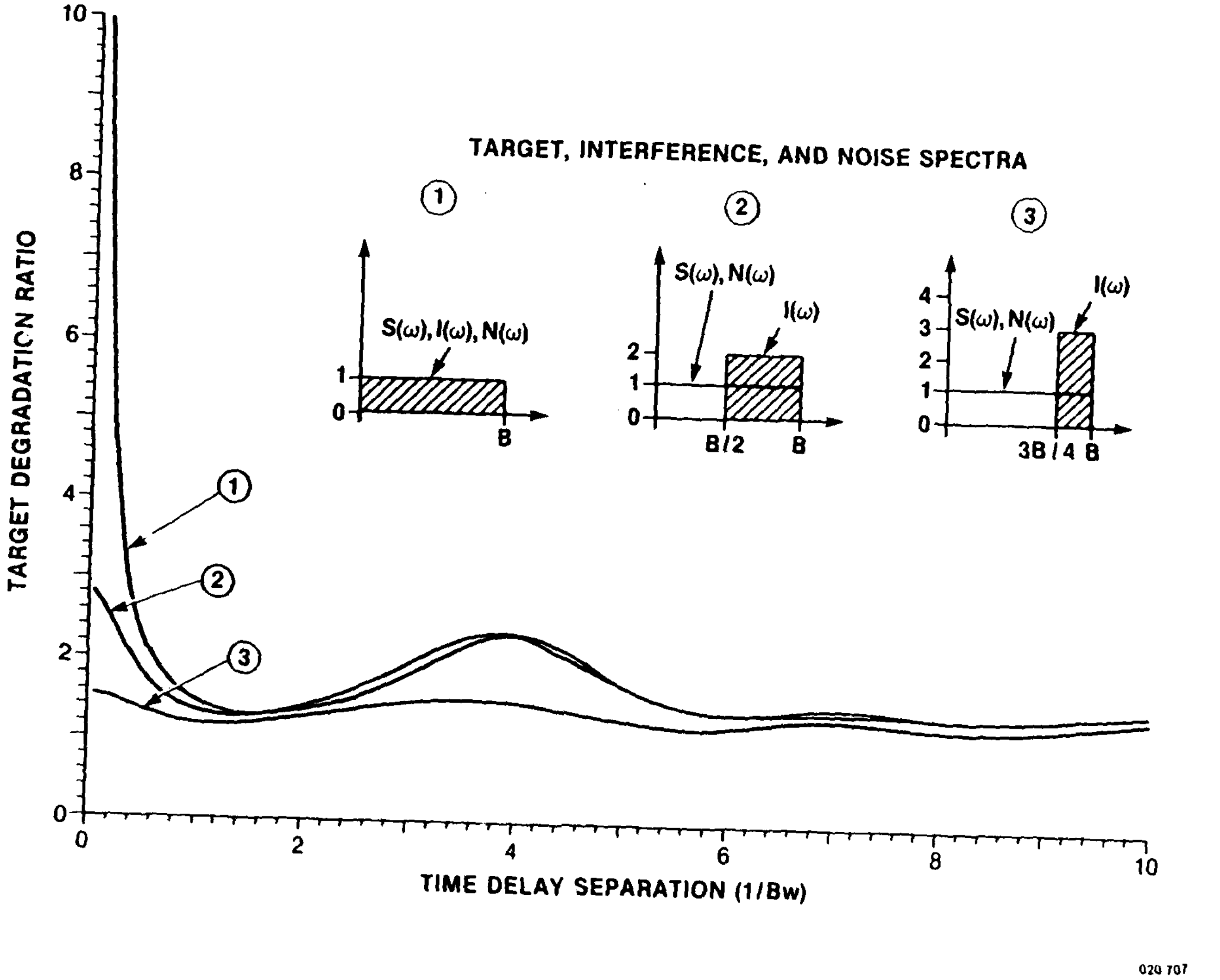

Figure 3-7. Two-Sensor, Two-Target CRLB Degradation Ratio Versus Time Delay Separation and Interference Spectrum (SNR = 0 dB, INR = 0 dB)

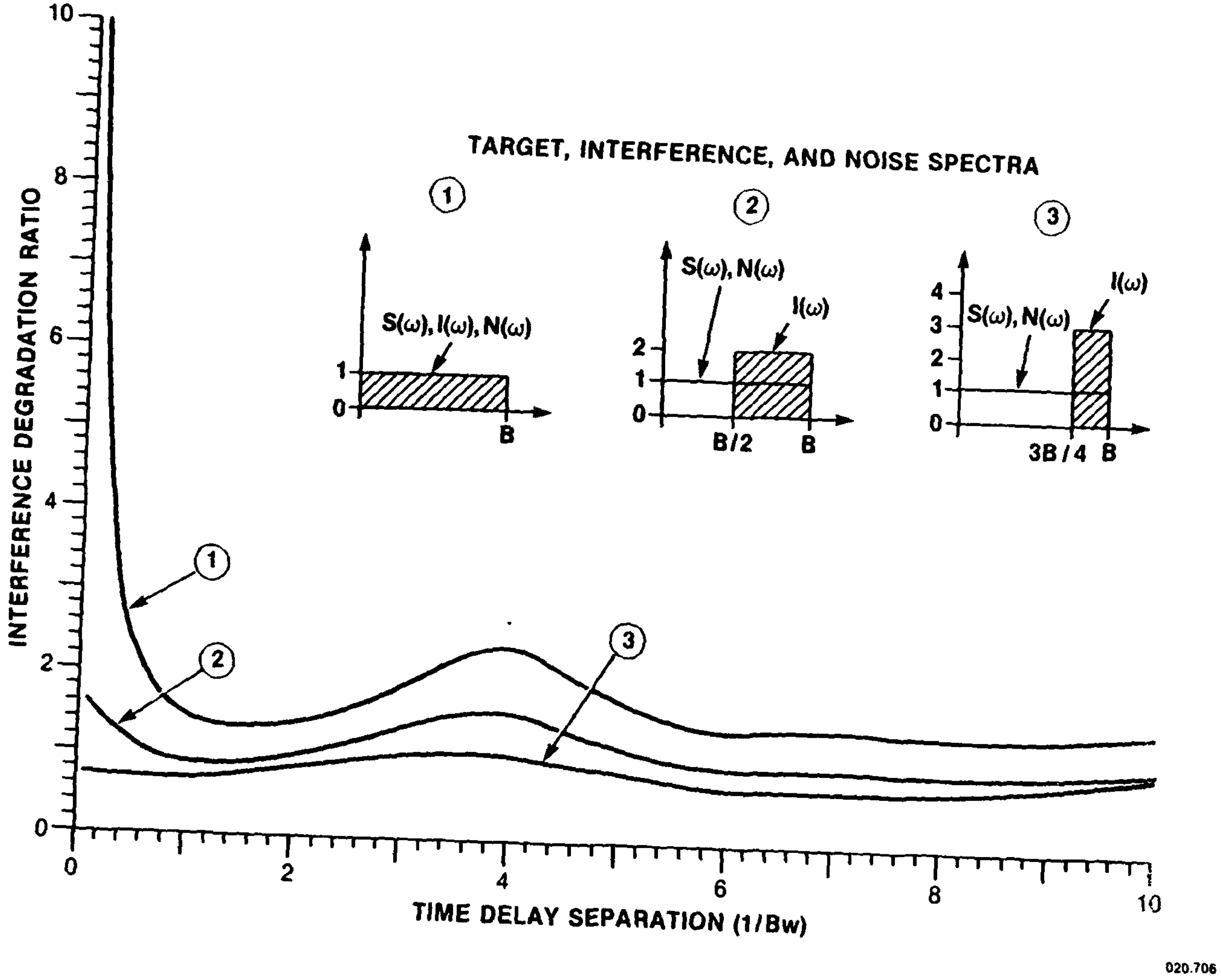

Figure 3-8. Two-Sensor, Two-Target CRLB Degradation Ratio Versus Time Delay Separation and Interference Spectrum (SNR = 0 dB, INR = 0 dB)

For convenience we select the intersensor time delay as the minimum time delay set. If we denote this parameter set by $\underline{\theta} = [\tau_1, \tau_2, \ldots, \tau_{M-1}]^T$, then the optimum estimate of this time delay set is given by the M-1 likelihood equation:

$$\frac{\partial\Lambda}{\partial\tau_i} = \int_0^\infty j\omega \left[|\tilde{h}_i|^2 \; \underline{\alpha}^* \; \tilde{Q}_i^{-1} \; V_i(\Phi_i - \tilde{b}_i \; 1_M) \; V_i^* \; Q_i^{-1} \; \underline{\alpha} - T\tilde{b}_i \right] \frac{d\omega}{2\pi} = 0 \tag{3.5.1-28}$$

for i = 1, 2, ..., M-1.

It was derived in Appendix C that the CRLB of time delay estimate for every inter-sensor time delay is identical and is given by

$$\mathrm{VAR}(\hat{\tau}_i) \geq \left[M \sum_{k=1}^{B} \omega_k^2 \; \frac{S_k^2/N_k^2}{1 + M\, S_k/N_k} \right]^{-1} \; ; \qquad i = 1, 2, \ldots, M-1 \tag{3.5.1-29a}$$

or in continuous form

$$\mathrm{VAR}(\hat{\tau}_i) \geq 2\pi \left[MT \int_0^\infty \frac{S^2(\omega)/N^2(\omega)}{1 + M\, S(\omega)/N(\omega)} \; \omega^2 \, d\omega \right]^{-1} \tag{3.5.1-29b}$$

Thus, the CRLB of any two adjacent sensors improves with increased M, the total number of sensors. Furthermore, Appendix C shows that the time delay estimate between any two sensors also has the same CRLB. In addition, the covariance of any two time delay estimates is

$$\mathrm{COV}(\hat{\tau}_i, \hat{\tau}_j) = \begin{cases} -\frac{1}{2}\,\mathrm{VAR}(\hat{\tau}_i) \; ; & \text{if } |i-j| = 1 \\ 0 \; ; & \text{if } |i-j| > 1 \end{cases} \tag{3.5.1-30a}$$

In other words, the correlation coefficient is

$$\rho_{ij} = \frac{\mathrm{COV}(\hat{\tau}_i, \hat{\tau}_j)}{\left[\mathrm{VAR}(\hat{\tau}_i)\ \mathrm{VAR}(\hat{\tau}_j)\right]^{\frac{1}{2}}}$$

$$= \begin{cases} -\frac{1}{2} & ;\ \text{if } |i - j| = 1 \\ 0 & ;\ \text{if } |i - j| > 1 \end{cases} \qquad (3.5.1\text{-}30b)$$

Thus far, we have studied only the optimal time delay processing of a multisensor array system. The objective of a passive sonar system, however, is to localize an acoustic source of interest (i.e., obtaining its range and bearing). For passive localization, a minimum of three sensors (assuming omni-directional response) is needed. However, for a far field assumption, two sensors can yield good bearing information. In the following section, we study in some detail the optimum processor structure for range and bearing estimations from a three-sensor array.

3.5.2 Localization Parameter Estimation

In this section we examine in detail the optimum processor structure for estimating localization parameters (i.e., range and bearing). We compare the performance of the optimum processor to the conventional approach, where range and bearing are obtained from the measurement of two time delay pairs. We discuss briefly the optimality of the direct range and bearing estimation approach and of the indirect time delay estimation approach.

Consider a general three-sensor array geometry shown in Figure 3-9. We are interested in obtaining an optimum estimation of range and bearing using outputs from a three-sensor array. From Appendix D, we note that the range and bearing information is contained in the two incremental time delays. Let τ_1 denote the time delay between sensor 1 and 2, and let τ_2 denote the time delay between sensors 2 and 3. Then one can write

$$\tau_1 = D_2 - D_1 = \frac{r - \sqrt{r^2 + L_1^2 - 2rL_1 \cos(\pi - \theta)}}{c} \qquad (3.5.2\text{-}1)$$

$$\tau_2 = D_3 - D_2 = \frac{\sqrt{r^2 + L_2^2 - 2rL_2 \cos(\theta - \phi)} - r}{c} \qquad (3.5.2\text{-}2)$$

where D_i; i = 1, 2, and 3 are the propagation delays. The steering matrix is

$$V_k = \text{diag}\left\{ e^{-j\omega_k \tau_1} \quad 1 \quad e^{j\omega_k \tau_2} \right\} . \qquad (3.5.2\text{-}3)$$

Therefore, Equations (3.5.1-10a) and (3.5.1-10b) become

$$P_k = V_k \, 1_M \, V_k^* \qquad (3.5.2\text{-}4a)$$

$$\frac{\partial P_k}{\partial \theta_i} = j\omega_k \, V_k \, \frac{\partial \Phi}{\partial \theta_i} \, V_k^* \qquad (3.5.2\text{-}4b)$$

where θ_i is any time-delay related parameter of interest, and the matrix Φ is given by (see Equation 3.5.1-8d)

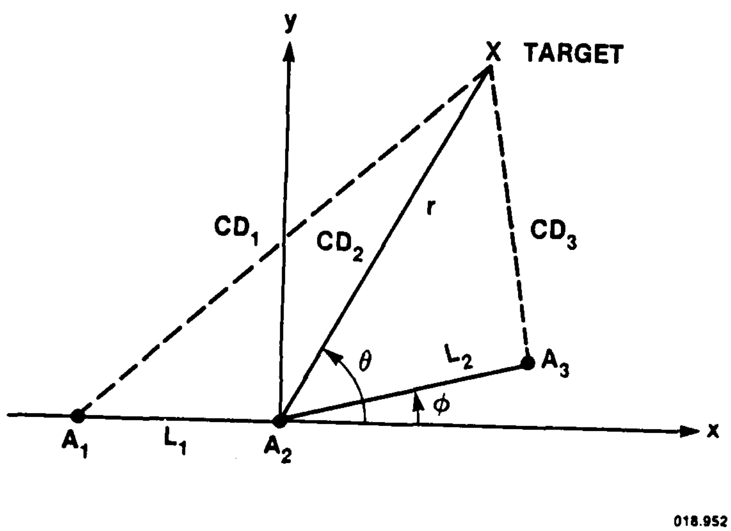

Figure 3-9. A Three-Sensor Array System

$$\Phi = \begin{bmatrix} 0 & -\tau_1 & -(\tau_1 + \tau_2) \\ \tau_1 & 0 & -\tau_2 \\ (\tau_1 + \tau_2) & \tau_2 & 0 \end{bmatrix} . \tag{3.5.2-4c}$$

The likelihood equations for range and bearing can be obtained from Equation (3.5.1-11a) as

$$\frac{\partial\Lambda(r,\theta)}{\partial r} = \sum_{k=1}^{B} j\omega_k \left[|\tilde{h}_k|^2 \underset{\sim}{\alpha}_k^* \tilde{Q}_k^{-1} V_k \left(\frac{\partial\Phi}{\partial r} - \tilde{b}_r 1_M \right) V_k^* \tilde{Q}_k^{-1} \underset{\sim}{\alpha}_k - \tilde{b}_r \right] = 0 \tag{3.5.2-5a}$$

and

$$\frac{\partial\Lambda(r,\theta)}{\partial\theta} = \sum_{k=1}^{B} j\omega_k \left[|\tilde{h}_k|^2 \underset{\sim}{\alpha}_k^* \tilde{Q}_k^{-1} V_k \left(\frac{\partial\Phi}{\partial\theta} - \tilde{b}_\theta 1_M \right) V_k^* \tilde{Q}_k^{-1} \underset{\sim}{\alpha}_k - \tilde{b}_\theta \right] = 0 \tag{3.5.2-5b}$$

where from Equation (3.5.1-11b) we have

$$\tilde{b}_r = \tilde{a}_k \, \mathrm{tr}\left(\tilde{Q}_k^{-1} V_k \frac{\partial\Phi}{\partial r} V_k^* \right) \tag{3.5.2-5c}$$

$$\tilde{b}_\theta = \tilde{a}_k \, \mathrm{tr}\left(\tilde{Q}_k^{-1} V_k \frac{\partial\Phi}{\partial\theta} V_k^* \right). \tag{3.5.2-5d}$$

For a single target in uncorrelated noise, it is easily verified that the biases are zero, i.e., and $b_r = b_\theta = 0$. Furthermore, for convenience we write $|\tilde{h}_k| = |h_k|$ and $\tilde{Q}_k^{-1} = I$, i.e., assuming spectrally identical but spatially uncorrelated noise power spectrum for each sensor. Therefore, the range and bearing likelihood equations reduce to

$$\frac{\partial\Lambda(r,\theta)}{\partial r} = \sum_{k=1}^{B} j\omega_k |h_k|^2 \underset{\sim}{\alpha}_k^* V_k \frac{\partial\Phi}{\partial r} V_k^* \underset{\sim}{\alpha}_k = 0 \tag{3.5.2-6a}$$

$$\frac{\partial\Lambda(r,\theta)}{\partial\theta} = \sum_{k=1}^{B} j\omega_k \, |h_k|^2 \, \underline{\alpha}_k^* \, V_k \, \frac{\partial\Phi}{\partial\theta} \, V_k^* \, \underline{\alpha}_k = 0 \, . \tag{3.5.2-6b}$$

where from Equation (3.5.1-4b) one obtains

$$|h_k|^2 = \frac{S_k/N_k^2}{1 + 3\, S_k/N_k} \, . \tag{3.5.2-6c}$$

Using the definitions of V_k, Φ and the resulting derivatives $\partial\Phi/\partial r$ and $\partial\Phi/\partial\theta$, the likelihood equations can be manipulated to yield (Appendix F)

$$\frac{\partial\Lambda(r,\theta)}{\partial r} = \frac{T}{2\pi}\frac{\partial}{\partial r}\int_{-\infty}^{\infty} |h(\omega)|^2 \, G(\omega;\, \tau_1,\, \tau_2)\, d\omega = 0 \tag{3.5.2-7a}$$

$$\frac{\partial\Lambda(r,\theta)}{\partial\theta} = \frac{T}{2\pi}\frac{\partial}{\partial\theta}\int_{-\infty}^{\infty} |h(\omega)|^2 \, G(\omega;\, \tau_1,\, \tau_2)\, d\omega = 0 \tag{3.5.2-7b}$$

where

$$G\,(\omega;\, \tau_1,\, \tau_2) = G_{21}(\omega)e^{j\omega\tau_1} + G_{32}(\omega)e^{j\omega\tau_2} + G_{31}(\omega)e^{j\omega(\tau_1+\tau_2)} \tag{3.5.2-7c}$$

and $G_{ij}(\omega) = T\alpha_i\alpha_j^*$ is the estimated cross-power spectrum between sensor i and j. Now we define the joint parameter ambiguity function by

$$R\,(\tau_1,\, \tau_2) = \int_{-\infty}^{\infty} |h(\omega)|^2 \, G(\omega;\, \tau_1,\, \tau_2)\, \frac{d\omega}{2\pi}$$

$$= \int_{-\infty}^{\infty} |h(\omega)|^2 \left[G_{21}(\omega)e^{j\omega\tau_1} + G_{32}(\omega)e^{j\omega\tau_2} + G_{31}(\omega)e^{j\omega(\tau_1+\tau_2)}\right] \frac{d\omega}{2\pi}$$

$$= R_{21}(\tau_1) + R_{32}(\tau_2) + R_{31}(\tau_1 + \tau_2) \ . \tag{3.5.2-8}$$

Therefore, the optimum (ML) estimate of range and bearing is obtained by seeking the simultaneous nulls of the likelihood equations:

$$\frac{\partial\Lambda(r,\theta)}{\partial r} = T \frac{\partial}{\partial r} R(\tau_1, \tau_2) = 0 \tag{3.5.2-9a}$$

$$\frac{\partial\Lambda(r,\theta)}{\partial\theta} = T \frac{\partial}{\partial\theta} R(\tau_1, \tau_2) = 0 \ . \tag{3.5.2-9b}$$

Now relate the time delays to range and bearing by

$$\begin{bmatrix} r \\ \theta \end{bmatrix} = \begin{bmatrix} f_1(\tau_1, \tau_2) \\ f_2(\tau_1, \tau_2) \end{bmatrix} = \underline{f}(\tau_1, \tau_2) \tag{3.5.2-9c}$$

and its inverse by

$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \underline{f}^{-1}(r,\theta) \ . \tag{3.5.2-9d}$$

Then the joint estimation of range and bearing can be realized by an open loop processor described in Figure 3-10. In the literature, this is known as a focused beamformer because the optimum estimate of range and bearing is obtained when the beam is focused on the target.

Note that the optimum estimate of range and bearing (r^*, θ^*) must correspond to the time delay pair (τ_1^*, τ_2^*) which defines the peak of the joint time delay ambiguity function $R(\tau_1, \tau_2)$. Therefore, an equivalent theoretical approach is to first seek the optimum time delay pair (τ_1^*, τ_2^*) and then transform it to the corresponding range and bearing. For practical applications, however, it is sometimes more convenient and simpler to search and track in the time delay parameter than the range and bearing parameters since the ambiguity function is symmetrical and uni-modal (for high SNR) in the time delay variables. More importantly, for a low SNR environment where a T-second

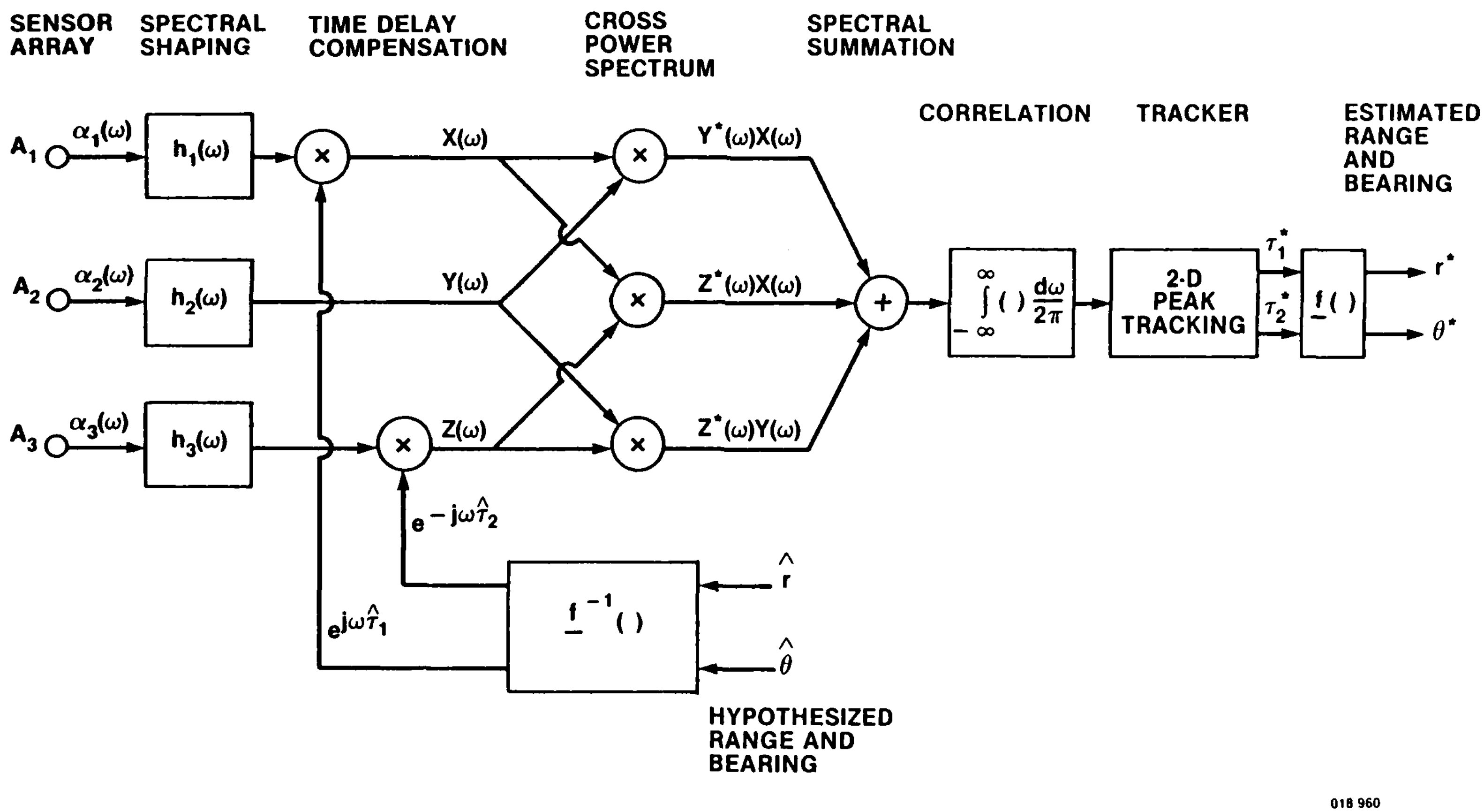

Figure 3-10. Optimum Range and Bearing Estimation from Three-Sensor Arrays (Focused Beamformer)

observation MLE processor fails to be efficient, the symmetric property of the time delay ambiguity function allows a simple tracking filter design which could provide optimum estimates by increasing the effective coherent integration time. Unfortunately, the resulting non-linear transformation from time delays to range and bearing for the practical approach renders the otherwise Gaussian noise process to be non-Gaussian.

Note the focused beamformer implementation shown in Figure 3-10 requires a two-dimensional (2-D) peak detector. However, a time delay approach using only two 1-D null detectors can be realized without any loss of performance. This can be seen as follows. Rewriting Equations (3.5.2-9a and b) in terms of the time delay variables and using the chain rule of differentiation yields

$$\frac{\partial \Lambda(r,\theta)}{\partial r} = T\left[\frac{\partial R(\tau_1, \tau_2)}{\partial \tau_1} \frac{\partial \tau_1}{\partial r} + \frac{\partial R(\tau_1, \tau_2)}{\partial \tau_2} \frac{\partial \tau_2}{\partial r}\right] = 0 \qquad (3.5.2\text{-}10a)$$

$$\frac{\partial \Lambda(r,\theta)}{\partial \theta} = T\left[\frac{\partial R(\tau_1, \tau_2)}{\partial \tau_1} \frac{\partial \tau_1}{\partial \theta} + \frac{\partial R(\tau_1, \tau_2)}{\partial \tau_2} \frac{\partial \tau_2}{\partial \theta}\right] = 0\ . \qquad (3.5.2\text{-}10b)$$

Using the simple relations

$$\tau_1 = -\frac{L_1}{c}\cos\theta - \frac{L_1^2 \sin^2\theta}{2rc} \quad \text{and}$$

$$\tau_2 = -\frac{L_2}{c}\cos\theta + \frac{L_2^2 \sin^2\theta}{2rc}$$

as shown in Equations (D-2a) and (D-2b), respectively in Appendix D, we observe that the derivatives of τ_1 and τ_2 w.r.t. range are zero only for infinite range and the derivatives w.r.t. bearing are zero only at the endfire direction. Therefore, for all other situations, the necessary and sufficient conditions for Equations (3.5.2-10a) and (3.5.2-10b) to be true are

$$\frac{\partial R(\tau_1,\ \tau_2)}{\partial \tau_1} = 0 \qquad (3.5.2\text{-}10c)$$

$$\frac{\partial R(\tau_1,\ \tau_2)}{\partial \tau_2} = 0\ . \qquad (3.5.2\text{-}10d)$$

Thus, using Equation (3.5.2-8) in (3.5.2-10a-d) yields

$$\frac{\partial R(\tau_1,\ \tau_2)}{\partial \tau_1} = T\,\frac{\partial}{\partial \tau_1}\left[R_{21}(\tau_1) + R_{31}(\tau_1 + \tau_2)\right]$$

$$= \frac{T}{2\pi}\,\frac{\partial}{\partial \tau_1}\int_{-\infty}^{\infty} |h(\omega)|^2 \left[G_{21}(\omega)e^{j\omega\tau_1} + G_{31}(\omega)e^{j\omega(\tau_1+\tau_2)}\right] d\omega \qquad (3.5.2\text{-}10e)$$

and

$$\frac{\partial R(\tau_1,\ \tau_2)}{\partial \tau_2} = T\,\frac{\partial}{\partial \tau_2}\left[R_{32}(\tau_2) + R_{31}(\tau_1 + \tau_2)\right]$$

$$= \frac{T}{2\pi}\,\frac{\partial}{\partial \tau_2}\int_{-\infty}^{\infty} |h(\omega)|^2 \left[G_{32}(\omega)e^{j\omega\tau_2} + G_{31}(\omega)e^{j\omega(\tau_1+\tau_2)}\right] d\omega \qquad (3.5.2\text{-}10f)$$

where we have used the fact that

$$\frac{\partial R_{32}(\tau_2)}{\partial \tau_1} = \frac{\partial R_{21}(\tau_1)}{\partial \tau_2} = 0\ . \qquad (3.5.2\text{-}10g)$$

An optimum three-sensor array processor using two correlators is shown in Figure 3-11.

We next calculate the CRLB for the optimum three-sensor array processor. First we calculate the CRLB for the optimum time delay estimate and compare it to the conventional approach. Then the CRLB for the localization parameters is obtained by relating it to the time delay estimate.

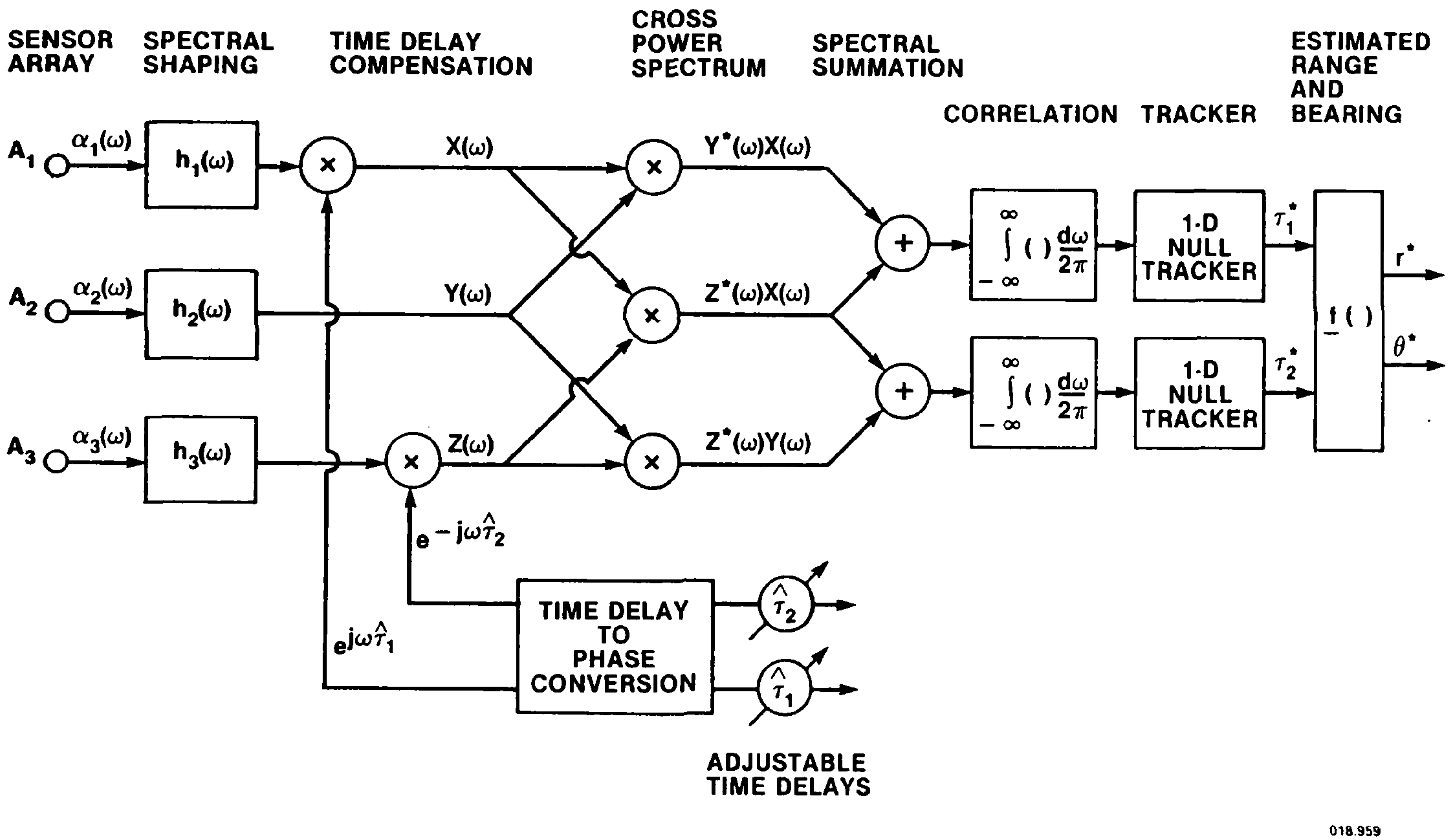

Figure 3-11. Optimum Range and Bearing Estimation from Three-Sensor Arrays Using Two Correlators

From Appendix C, we obtain the elements of the Fisher Information matrix for optimum one-target three-sensor time delay estimation as:

$$J_{ij} = -\mathrm{tr}\left(\frac{\partial\Phi}{\partial\tau_i}\frac{\partial\Phi}{\partial\tau_j}\right)\sum_{k=1}^{B}\omega_k^2\left(\frac{S_k^2/N_k^2}{1 + 3\ S_k/N_k}\right). \qquad (3.5.2\text{-}11)$$

From Equation (3.5.2-4c) we find

$$\frac{\partial\Phi}{\partial\tau_1} = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} ; \quad \frac{\partial\Phi}{\partial\tau_2} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix} \qquad (3.5.2\text{-}12a)$$

and therefore

$$\mathrm{tr}\left(\frac{\partial\Phi}{\partial\tau_1}\frac{\partial\Phi}{\partial\tau_1}\right) = -4 \qquad (3.5.2\text{-}12b)$$

$$\mathrm{tr}\left(\frac{\partial\Phi}{\partial\tau_1}\frac{\partial\Phi}{\partial\tau_2}\right) = -2 \qquad (3.5.2\text{-}12c)$$

and

$$\mathrm{tr}\left(\frac{\partial\Phi}{\partial\tau_2}\frac{\partial\Phi}{\partial\tau_2}\right) = -4\ . \qquad (3.5.2\text{-}12d)$$

From Appendix B, the CRLB for a two-parameter joint estimation is:

$$\mathrm{VAR}(\hat{\tau}_1) \geq \frac{1}{(1 - M_{12}^2)}\ \frac{1}{J_{11}} \qquad (3.5.2\text{-}13a)$$

$$\mathrm{VAR}(\hat{\tau}_2) \geq \frac{1}{(1 - M_{12}^2)}\ \frac{1}{J_{22}} \qquad (3.5.2\text{-}13b)$$

where M_{12}, the coefficient of mutual dependence, is given by

$$M_{12} = \frac{J_{12}}{(J_{11}\ J_{22})^{1/2}} \ . \qquad (3.5.2\text{-}13c)$$

Now using Equations (3.5.2-11) and (3.5.2-12a-d) or Equation (3.5.1-29a), we obtain

$$\mathrm{VAR}(\hat{\tau}_1) = \mathrm{VAR}(\hat{\tau}_2) \geq \left[3 \sum_{k=1}^{B} \omega_k^2 \left(\frac{S_k^2/N_k^2}{1 + 3\ S_k\ N_k}\right)\right]^{-1} \ . \qquad (3.5.2\text{-}14)$$

Note that for the general case of M sensors it was shown in Appendix C that the CRLB of the incremental time delay estimates is given by

$$\mathrm{VAR}(\hat{\tau}_i) \geq \left[M \sum_{k=1}^{B} \omega_k^2 \left(\frac{S_k^2/N_k^2}{1 + M\ S_k/N_k}\right)\right]^{-1} ; \qquad i = 1, 2, \ldots, M\text{-}1 \ . \qquad (3.5.2\text{-}15)$$

Thus, the time delay estimate improves with the increased number of spatial sensor observations.

We now discuss the conventional approach to estimating range and bearing. A conventional three-sensor array signal processor is shown in Figure 3-12. In the conventional approach, only two cross-power spectra are processed. For this case, the CRLB of the time delay estimate is given by (see Equation (3.5.1-16a))

$$\mathrm{VAR}_c(\hat{\tau}_1) = \mathrm{VAR}_c(\tau_2) \geq \left[2 \sum_{k=1}^{B} \omega_k^2 \left(\frac{S_k^2/N_k^2}{1 + 2\ S_k/N_k}\right)\right]^{-1} \ . \qquad (3.5.2\text{-}16a)$$

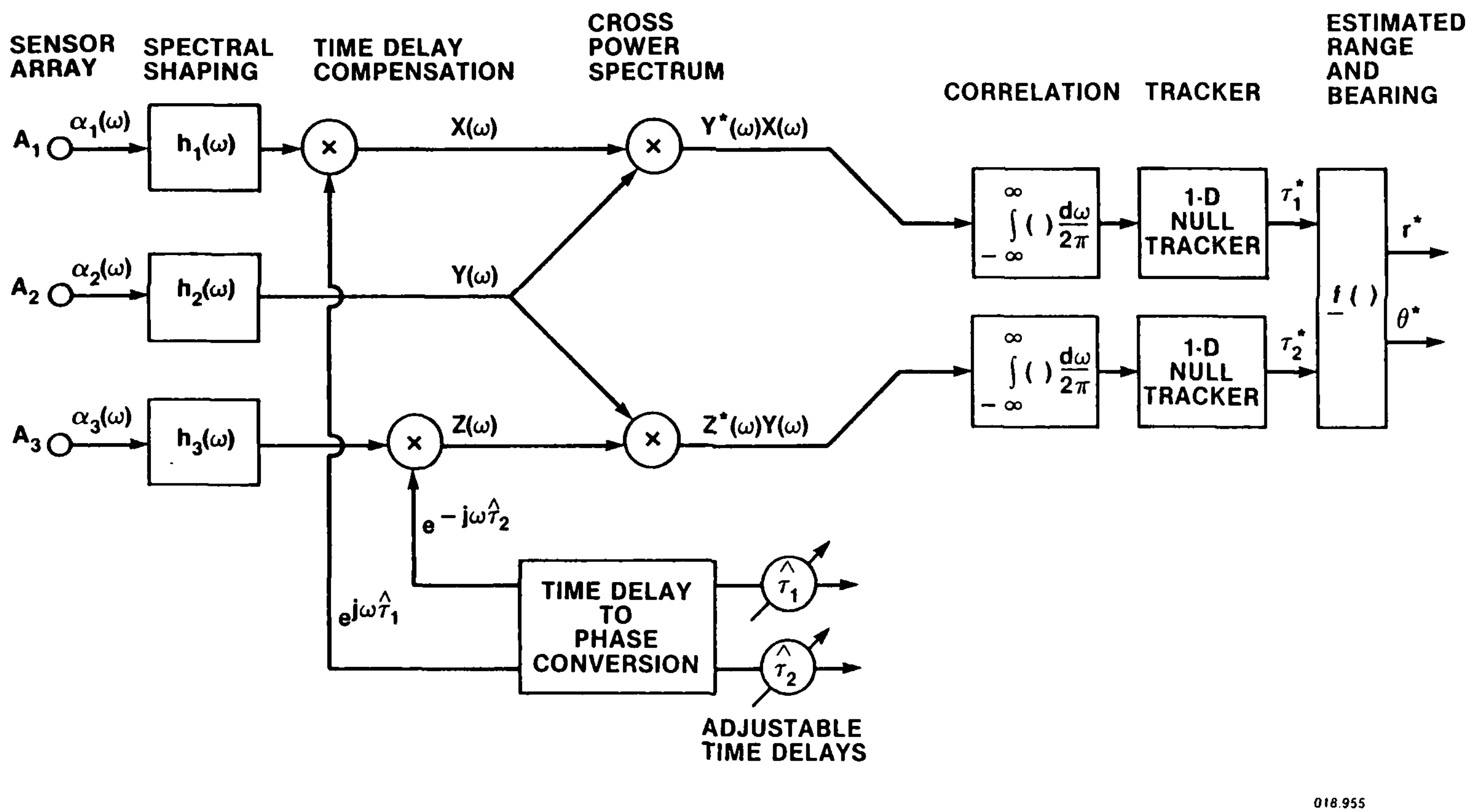

Figure 3-12. Conventional Range and Bearing Estimation from Three-Sensor Arrays Using Two Correlators

Therefore, the time delay variance ratio of the optimum to the conventional is

$$\frac{VAR(\hat{\tau}_1)}{VAR_c(\hat{\tau}_1)} = \frac{2}{3}\,\frac{\displaystyle\sum_{k=1}^{B} \omega_k^2 \left(\frac{S_k^2/N_k^2}{1 + 2\ S_k/N_k}\right)}{\displaystyle\sum_{k=1}^{B} \omega_k^2 \left(\frac{S_k^2/N_k^2}{1 + 3\ S_k/N_k}\right)} \tag{3.5.2-16b}$$

and for a flat signal and noise power spectra, Equation (3.5.2-16b) is simplified to

$$\frac{VAR(\hat{\tau}_1)}{VAR_c(\hat{\tau}_1)} = \left(\frac{2}{3}\right)\left(\frac{1 + 3\ S/N}{1 + 2\ S/N}\right) . \tag{3.5.2-16c}$$

Thus, at low SNR, the CRLB of the optimum processor is improved by a factor of 2/3 (1.8 dB) with respect to the conventional approach.

Because the two time delays are correlated due to common noise channels, a more meaningful approach is to compare the one sigma error ellipse area between the optimum and the conventional. From Appendices C and E, we obtain the two time delay covariance matrices as

$$P_o = \begin{bmatrix} 1 & \frac{-1}{2} \\ \frac{-1}{2} & 1 \end{bmatrix} VAR(\hat{\tau}_1) \tag{3.5.2-16d}$$

$$P_c = \begin{bmatrix} 1 & \frac{-S/N}{1 + 2\ S/N} \\ \frac{-S/N}{1 + 2\ S/N} & 1 \end{bmatrix} VAR_c(\hat{\tau}_1) \tag{3.5.2-16e}$$

where P_o and P_c denote, respectively, the optimum and conventional covariance matrices. Therefore, the ratio of their area is

$$\frac{A_o}{A_c} = \frac{\pi |P_o|^{\frac{1}{2}}}{\pi |P_c|^{\frac{1}{2}}}$$

$$= \sqrt{\frac{3}{4} \frac{(1 + 2\ S/N)^2}{(1 + 3\ S/N)(1 + S/N)} \quad \frac{VAR(\hat{\tau}_1)}{VAR_c(\hat{\tau}_1)}} \ . \qquad (3.5.2\text{-}16f)$$

Using Equations (3.5.2-16d-e) in Equation (3.5.2-16f) yields the desired result:

$$\frac{A_o}{A_c} = \sqrt{\frac{1}{3} \frac{1 + 3\ S/N}{1 + S/N}} \ . \qquad (3.5.2\text{-}16g)$$

Thus Equation (3.5.2-16g) states that in a low SNR environment ($S/N << 1$), the error ellipse of the optimum approach has an area equal approximately to one half the conventional approach. This represents a substantial loss in performance by the conventional approach. On the other hand, in a high SNR environment ($S/N >> 1$) the conventional processor approaches the optimum.

The CRLB on the localization parameters for the focused beamformer can be obtained as follows. Defining the Fisher Information matrix by

$$F = \begin{bmatrix} F_{rr} & F_{r\theta} \\ F_{\theta r} & F_{\theta\theta} \end{bmatrix} ,$$

then the CRLB for the range and bearing estimates is:

$$VAR(\hat{r}) = \frac{1}{(1 - M_{12}^2)} \frac{1}{F_{rr}} \qquad (3.5.2\text{-}17a)$$

$$\mathrm{VAR}(\hat{\theta}) = \frac{1}{(1 - M_{12}^2)} \frac{1}{F_{\theta\theta}} \tag{3.5.2-17b}$$

where

$$M_{12} = \frac{F_{r\theta}}{(F_{rr}\, F_{\theta\theta})^{\frac{1}{2}}} \, . \tag{3.5.2-17c}$$

Again from Appendix C, we find:

$$F_{rr} = -\mathrm{tr}\left(\frac{\partial\Phi}{\partial r}\;\frac{\partial\Phi}{\partial r}\right) \sum_{k=1}^{B} \omega_k^2 \left(\frac{S_k^2/N_k^2}{1 + 3\;S_k/N_k}\right) \tag{3.5.2-18a}$$

$$F_{r\theta} = -\mathrm{tr}\left(\frac{\partial\Phi}{\partial r}\,\frac{\partial\Phi}{\partial \theta}\right) \sum_{k=1}^{B} \omega_k^2 \left(\frac{S_k^2/N_k^2}{1 + 3\;S_k/N_k}\right) \tag{3.5.2-18b}$$

and

$$F_{\theta\theta} = -\mathrm{tr}\left(\frac{\partial\Phi}{\partial \theta}\,\frac{\partial\Phi}{\partial \theta}\right) \sum_{k=1}^{B} \omega_k^2 \left(\frac{S_k^2/N_k^2}{1 + 3\;S_k/N_k}\right) . \tag{3.5.2-18c}$$

Now

$$\mathrm{tr}\left(\frac{\partial\Phi}{\partial r}\,\frac{\partial\Phi}{\partial \theta}\right) = \mathrm{tr}\left[\left(\sum_{i=1}^{2} \frac{\partial\Phi}{\partial\tau_i}\,\frac{\partial\tau_i}{\partial r}\right)\left(\sum_{j=1}^{2} \frac{\partial\Phi}{\partial\tau_j}\,\frac{\partial\tau_j}{\partial \theta}\right)\right]$$

$$= \sum_{i=1}^{2} \sum_{j=i}^{2} \left(\frac{\partial\tau_i}{\partial r}\,\frac{\partial\tau_j}{\partial\theta}\right) \;\mathrm{tr}\left(\frac{\partial\Phi}{\partial\tau_i}\,\frac{\partial\Phi}{\partial\tau_j}\right) \tag{3.5.2-19a}$$

and using Equations (3.5.2-12a-d), we obtain

$$\operatorname{tr}\left(\frac{\partial\Phi}{\partial r}\frac{\partial\Phi}{\partial\theta}\right) = -4\left(\frac{\partial\tau_1}{\partial r}\frac{\partial\tau_1}{\partial\theta} + \frac{\partial\tau_1}{\partial r}\frac{\partial\tau_2}{\partial\theta} + \frac{\partial\tau_2}{\partial r}\frac{\partial\tau_2}{\partial\theta} + \frac{\partial\tau_2}{\partial r}\frac{\partial\tau_1}{\partial\theta}\right) \qquad (3.5.2\text{-}19b)$$

and similarly

$$\operatorname{tr}\left(\frac{\partial\Phi}{\partial r}\frac{\partial\Phi}{\partial r}\right) = -4\left[\left(\frac{\partial\tau_1}{\partial r}\right)^2 + \frac{\partial\tau_1}{\partial r}\frac{\partial\tau_2}{\partial r} + \left(\frac{\partial\tau_2}{\partial r}\right)^2\right] \qquad (3.5.2\text{-}19c)$$

$$\operatorname{tr}\left(\frac{\partial\Phi}{\partial\theta}\frac{\partial\Phi}{\partial\theta}\right) = -4\left[\left(\frac{\partial\tau_1}{\partial\theta}\right)^2 + \frac{\partial\tau_1}{\partial\theta}\frac{\partial\tau_2}{\partial\theta} + \left(\frac{\partial\tau_2}{\partial\theta}\right)^2\right]. \qquad (3.5.2\text{-}19d)$$

Using Equations (3.5.2-1) and (3.5.2-2) and evaluating the derivatives at the true range and bearing (R, B), we obtain the following:

$$\left.\frac{\partial\tau_1}{\partial r}\right|_{R,B} = \frac{1}{2c}\left(\frac{L_1 \sin B}{R}\right)^2 \qquad (3.5.2\text{-}20a)$$

$$\left.\frac{\partial\tau_2}{\partial r}\right|_{R,B} = -\frac{1}{2}\left(\frac{L_2 \sin(B-\phi)}{R}\right)^2 \qquad (3.5.2\text{-}20b)$$

$$\left.\frac{\partial\tau_1}{\partial\theta}\right|_{R,B} = \frac{L_1}{c}\sin B - \frac{L_1^2}{c}\left(\frac{\sin 2B}{R}\right) \qquad (3.5.2\text{-}20c)$$

and

$$\left.\frac{\partial\tau_2}{\partial\theta}\right|_{R,B} = \frac{L_2}{c}\sin(B-\phi) + \frac{L_2^2}{2c}\left(\frac{\sin 2(B-\phi)}{R}\right). \qquad (3.5.2\text{-}20d)$$

For simplicity we assume a special case where the sensor arrays are co-linear with equal separation ($L = L_1 = L_2$) and the target is at broadside. Thus, substituting Equations (3.5.2-20a-d) in Equations (3.5.2-9b-d) yields:

$$\operatorname{tr}\left(\frac{\partial \Phi}{\partial r}\frac{\partial \Phi}{\partial \theta}\right)\Bigg|_{R,B} = 0 \tag{3.5.2-21a}$$

$$\operatorname{tr}\left(\frac{\partial \Phi}{\partial r}\frac{\partial \Phi}{\partial r}\right)\Bigg|_{R,B} = \frac{-1}{c^2}\left(\frac{L}{R}\right)^4 \tag{3.5.2-21b}$$

and

$$\operatorname{tr}\left(\frac{\partial \Phi}{\partial \theta}\frac{\partial \Phi}{\partial \theta}\right)\Bigg|_{R,B} = -12\left(\frac{L}{c}\right)^2 \quad . \tag{3.5.2-21c}$$

Therefore, the CRLB of range and bearing evaluated at the true range and bearing is given by:

$$\mathrm{VAR}(\hat{r}) \geq \frac{1}{(1 - M_{12}^2)} \left\{-\operatorname{tr}\left[\left(\frac{\partial \Phi}{\partial r}\right)^2\right] \sum_{k=1}^{B} \omega_k^2 \left(\frac{S_k^2/N_k^2}{1 + 3\, S_k/N_k}\right)\right\}^{-1} \tag{3.5.2-22a}$$

$$\mathrm{VAR}(\hat{\theta}) \geq \frac{1}{(1 - M_{12}^2)} \left\{-\operatorname{tr}\left[\left(\frac{\partial \Phi}{\partial \theta}\right)^2\right] \sum_{k=1}^{B} \omega_k^2 \left(\frac{S_k^2/N_k^2}{1 + 3\, S_k/N_k}\right)\right\}^{-1} \tag{3.5.2-22b}$$

And because of Equation (3.5.2-21a), range and bearing estimates are uncorrelated. Using Equation (3.5.2-14) these can be expressed in terms of the variance of the time delay estimate as follows:

$$\mathrm{VAR}(\hat{r}) \geq \frac{3\ \mathrm{VAR}(\hat{\tau}_1)}{-\mathrm{tr}\left(\frac{\partial\Phi}{\partial r}\right)^2} \tag{3.5.2-23a}$$

$$\mathrm{VAR}(\hat{\theta}) \geq \frac{3\ \mathrm{VAR}\ (\hat{\tau}_1)}{-\mathrm{tr}\left(\frac{\partial\Phi}{\partial\theta}\right)^2}\ . \tag{3.5.2-23b}$$

For the special case of a co-linear array and a broadside target, we find

$$\mathrm{VAR}(\hat{r}) \geq 3c^2 \left(\frac{R}{L}\right)^4 \mathrm{VAR}(\hat{\tau}_1) \tag{3.5.2-24a}$$

$$\mathrm{VAR}(\hat{\theta}) \geq \frac{1}{4}\left(\frac{c}{L}\right)^2 \mathrm{VAR}(\hat{\tau}_1)\ . \tag{3.5.2-24b}$$

Taking the ratio of Equation (3.5.2-24a) to Equation (3.5.2-24b) yields

$$\mathrm{VAR}(\hat{r}) \geq 12\ \frac{R^4}{L^2}\ \mathrm{VAR}(\hat{\theta})\ , \tag{3.5.2-24c}$$

which relates the range variance to the bearing variance. Thus, we obtain the well known results that the variance of the range estimate is proportional to the fourth power ratio of the true range to base line length, and the variance of the bearing estimate is inversely proportional to the square of the base length.

Note that the range and bearing variances of the focused beamformer approach (Equations (3.5.2-24a-b)) agree with Equations (G-15) and (G-16) of Appendix G, which were obtained via a geometric mapping from time delay measurements. Therefore, the one-step focused beamformer approach and the two-step time delay approach yield identical statistical performance.

Let S_o and S_c denote the area of the range/bearing one-sigma localization error ellipse for the optimum and the conventional approach, respectively. Then from Equations (3.5.2-24a) and (3.5.2-24b) one obtains

$$S_o = \pi\sqrt{VAR(\hat{r})\ VAR(\hat{\theta})}$$

$$= \frac{\sqrt{3}}{2}\left(\frac{\pi R^2 c^2}{L^3}\right) VAR(\hat{\tau}_1)\ . \qquad (3.5.2\text{-}25a)$$

On the other hand, for the conventional approach, it can be shown (Appendix G) that

$$S_c = \left(\frac{\pi R^2 c^2}{L^3}\right) \frac{\sqrt{(1 + 3\ S/N)(1 + S/N)}}{(1 + 2\ S/N)}\ VAR_c(\hat{\tau}_1)\ . \qquad (3.5.2\text{-}25b)$$

Therefore, using the relation given in Equation (3.2.5-16d), the ratio of the error ellipse area is

$$\frac{S_o}{S_c} = \sqrt{\frac{1}{3}\left(\frac{1 + 3\ S/N}{1 + S/N}\right)} \qquad (3.5.2\text{-}25c)$$

which has the same ratio as in time delay estimation (see Equation 3.5.2-16g). Thus Equation (3.5.2-25c) implies that the optimum processor yields a one-sigma localization error ellipse which is approximately one half ($1/\sqrt{3}$) that of the conventional approach in a low SNR environment. This improvement comes directly from a better bearing estimation using the optimum approach.* Note, it can be shown that the ranging performance is identical between the conventional and the optimum approach. (See Appendix G.)

*It was pointed out by Dr. J. Iannielo of the Naval Underwater Systems that optimum range and bearing estimation can also be achieved using the conventional approach. However, one must provide separate and different spectral shaping filters for range and bearing estimation.

3.5.3 Power Spectral Estimation

Our discussion thus far assumes that all target power spectra are known. This is one of the strongest assumptions we have made in studying the optimum signal processor. The resulting processors contain spectral shaping filters which are functions of the known target power spectra. In an actual implementation, the power spectra must be either known a priori or estimated. In this section we briefly examine the methodology of spectral estimation in a multisensor, multitarget environment and the relations between power spectral estimation and time delay estimation.

Thus, we seek the estimate $\hat{S}_{kj}$, the signal power spectral level of target j at frequency k. Recall from Equation (3.2-6) that the spectral levels are contained in the observation covariance matrix R_k. Let $\underline{S}$ be a column vector of all the unknown spectral levels. Thus, $\underline{S} = (S_{11}\ S_{21}\ .\ .\ .\ S_{B1};\ S_{12}\ S_{22}\ .\ .\ .\ S_{B2};\ .\ .\ .\ ;\ S_{1J}\ S_{2J}\ .\ .\ .\ S_{BJ})^T$ is a JB dimension vector. Therefore, from Equation (3.2-9) the likelihood equation for $\hat{S}_{kj}$ is

$$\frac{\partial \Lambda(\underline{S})}{\partial S_{kj}} = - \frac{\partial \Lambda_k(\underline{S})}{\partial S_{kj}} = 0; \qquad \begin{array}{l} j = 1, 2, . . . , J \\ k = 1, 2, . . . , B \end{array} \tag{3.5.3-1}$$

Note that the likelihood equations are decoupled in frequency. This implies that each equation can be solved separately. Now from Equation (3.2-10) we obtain

$$\frac{\partial \Lambda_k(\underline{S})}{\partial S_{kj}} = \mathrm{tr}\left(R_k^{-1} \frac{\partial R_k}{\partial S_{kj}}\right) + \underline{\alpha}_k^* \frac{\partial R_k^{-1}}{\partial S_{kj}} \underline{\alpha}_k \tag{3.5.3-2}$$

where R_k and R_k^{-1} are given by Equations (3.5.1-2a) and (3.5.1-3). Hence, the derivative of R_k and R_k^{-1} w.r.t. S_{kj} is given by

$$\frac{\partial R_k}{\partial S_{kj}} = P_{kj} = \underline{V}_{kj}\,\underline{V}_{kj}^{*} \tag{3.5.3-3a}$$

$$\frac{\partial R_k^{-1}}{\partial S_{kj}} = \frac{\partial}{\partial S_{kj}}\left(S_{kj}\,P_{kj} + \tilde{N}_{kj}\,\tilde{Q}_{kj}\right)^{-1}$$

$$= \frac{\partial}{\partial S_{kj}}\left(\frac{\tilde{Q}_{kj}^{-1}}{\tilde{N}_{kj}} - \frac{S_{kj}/\tilde{N}_{kj}^{2}}{1 + G_{kj}\,S_{kj}/\tilde{N}_{kj}}\,\tilde{Q}_{kj}^{-1}\,P_{kj}\,\tilde{Q}_{kj}^{-1}\right)$$

$$= -\left[\frac{1/\tilde{N}_{kj}^{2}}{1 + G_{kj}\,S_{kj}/\tilde{N}_{kj}} - \frac{G_{kj}\,S_{kj}/\tilde{N}_{kj}^{3}}{(1 + G_{kj}\,S_{kj}/\tilde{N}_{kj})^{2}}\right]\tilde{Q}_{kj}^{-1}\,P_{kj}\,\tilde{Q}_{kj}^{-1}$$

$$= -\frac{\tilde{Q}_{kj}^{-1}\,P_{kj}\,\tilde{Q}_{kj}^{-1}/\tilde{N}_{kj}^{2}}{(1 + G_{kj}\,S_{kj}/\tilde{N}_{kj})^{2}}\ . \tag{3.5.3-3b}$$

Therefore, the first term in Equation (3.5.3-2) can be written as

$$\mathrm{tr}\left(R_k^{-1}\,\frac{\partial R_k}{\partial S_{kj}}\right) = \mathrm{tr}\left(\frac{\tilde{Q}_{kj}^{-1}\,\underline{V}_{kj}\,\underline{V}_{kj}^{*}}{\tilde{N}_{kj}} - \frac{S_{kj}/\tilde{N}_{kj}^{2}}{1 + G_{kj}\,S_{kj}/\tilde{N}_{kj}}\,\tilde{Q}_{kj}^{-1}\,\underline{V}_{kj}\,\underline{V}_{kj}^{*}\,\tilde{Q}_{kj}^{-1}\,\underline{V}_{kj}\,\underline{V}_{kj}^{*}\right)$$

$$= \frac{G_{kj}}{\tilde{N}_{kj}} - \frac{S_{kj}/\tilde{N}_{kj}^{2}}{1 + G_{kj}\,S_{kj}/\tilde{N}_{kj}}\,G_{kj}^{2}$$

$$= \frac{G_{kj}/\tilde{N}_{kj}}{1 + G_{kj}\,S_{kj}/\tilde{N}_{kj}}\ . \tag{3.5.3-4}$$

Substituting Equations (3.5.3-3b) and (3.5.3-4) into (3.5.3-2), one obtains

$$\frac{\partial\Lambda(\underline{S})}{\partial S_{kj}} = - \frac{G_{kj}/\tilde{N}_{kj}}{1 + G_{kj}\ S_{kj}/\tilde{N}_{kj}} + \frac{|\underline{V}^{*}_{kj}\ \tilde{Q}^{-1}_{kj}\ \underline{\alpha}_{k}|^2}{(1 + G_{kj}\ S_{kj}/\tilde{N}_{kj})^2\ \tilde{N}^2_{kj}} = 0\ . \qquad (3.5.3\text{-}5)$$

Solving Equation (3.5.3-5) yields

$$\hat{S}_{kj} = \frac{|\underline{V}^{*}_{kj}\ \tilde{Q}^{-1}_{kj}\ \underline{\alpha}_{k}|^2}{G^2_{kj}} - \frac{\tilde{N}_{kj}}{G_{kj}}\ ; \qquad k = 1, 2, \ldots, B \quad j = 1, 2, \ldots, J\ . \qquad (3.5.3\text{-}6)$$

Thus, in order to estimate the J target power spectra, a total of JB equations must be solved if the optimum steering vector $\underline{V}^{*}_{kj}$ is known. If it is not known, then the J(M - 1) time delay equations must be solved simultaneously with the JB spectral equations. However, in practice the target spectrum can be modeled with a considerably smaller set of unknown parameters. Therefore, sampling the spectrum at appropriate frequencies should provide sufficient information to estimate the unknown spectral parameters.

Note that Equation (3.5.3-6) is an unbiased spectral estimator. This can be shown easily as follows.

Taking the expected value of Equation (3.5.3-6) yields

$$E(\hat{S}_{kj}) = \frac{1}{G^2_{kj}}\ E|\underline{V}^{*}_{kj}\ \tilde{Q}^{-1}_{kj}\ \underline{\alpha}_{k}|^2 - \frac{\tilde{N}_{kj}}{G_{kj}}$$

$$= \frac{1}{G^2_{kj}}\ \underline{V}^{*}_{kj}\ \tilde{Q}^{-1}_{kj}\ E(\underline{\alpha}_{k}\ \underline{\alpha}^{*}_{k})\ \tilde{Q}^{-1}_{kj}\ \underline{V}_{kj} - \frac{\tilde{N}_{kj}}{G_{kj}}\ . \qquad (3.5.3\text{-}7)$$

But from Equations (2.3-3a) and (3.2-7a):

$$E(\underline{\alpha}_k\ \underline{\alpha}_k^*) = S_{kj}\ \underline{V}_{kj}\ \underline{V}_{kj}^* + \tilde{N}_{kj}\ \tilde{Q}_{kj}\ . \tag{3.5.3-8}$$

Substituting Equation (3.5.3-8) into (3.5.3-7) yields

$$E(\hat{S}_{kj}) = S_{kj} + \frac{N_{kj}\ \underline{V}_{kj}^*\ \tilde{Q}_{kj}^{-1}\ \tilde{Q}_{kj}\ \tilde{Q}_{kj}^{-1}\ \underline{V}_{kj}}{G_{kj}^2} - \frac{\tilde{N}_{kj}}{G_{kj}}\ .$$

$$= S_{kj} \tag{3.5.3-9}$$

Next we briefly examine the optimum spectral estimator for the one-target and two-target cases.

3.5.3.1 Case 1: One Target, M Sensors (J=1, M=M). For a single target, we have $\hat{N}_{kj} = N_k$, $\hat{Q}_{kj}^{-1} = Q_k$, and $G_{kj} = G_k = \underline{V}_k^*\ Q_k^{-1}\ \underline{V}_k$. Equation (3.5.3-6) becomes

$$\hat{S}_k = \frac{|\underline{V}_k^*\ Q_k^{-1}\ \underline{\alpha}_k|^2}{G_k^2} - \frac{N_k}{G_k}\ . \tag{3.5.3-10}$$

Furthermore, by letting $Q_k = I$ (i.e., noises are uncorrelated from sensor to sensor), then Equation (3.5.3-10) can be simplified to

$$\hat{S}_k = \frac{|\underline{V}_k^*\ \underline{\alpha}_k|^2}{M^2} - \frac{N_k}{M}\ . \tag{3.5.3-11}$$

Figure 3-13 shows the optimum one-target multisensor spectral estimator.

3.5.3.2 Case 2: Two Targets, M Sensors (J=2, M=M). For the two-target case, Equation (3.5.3-6) becomes

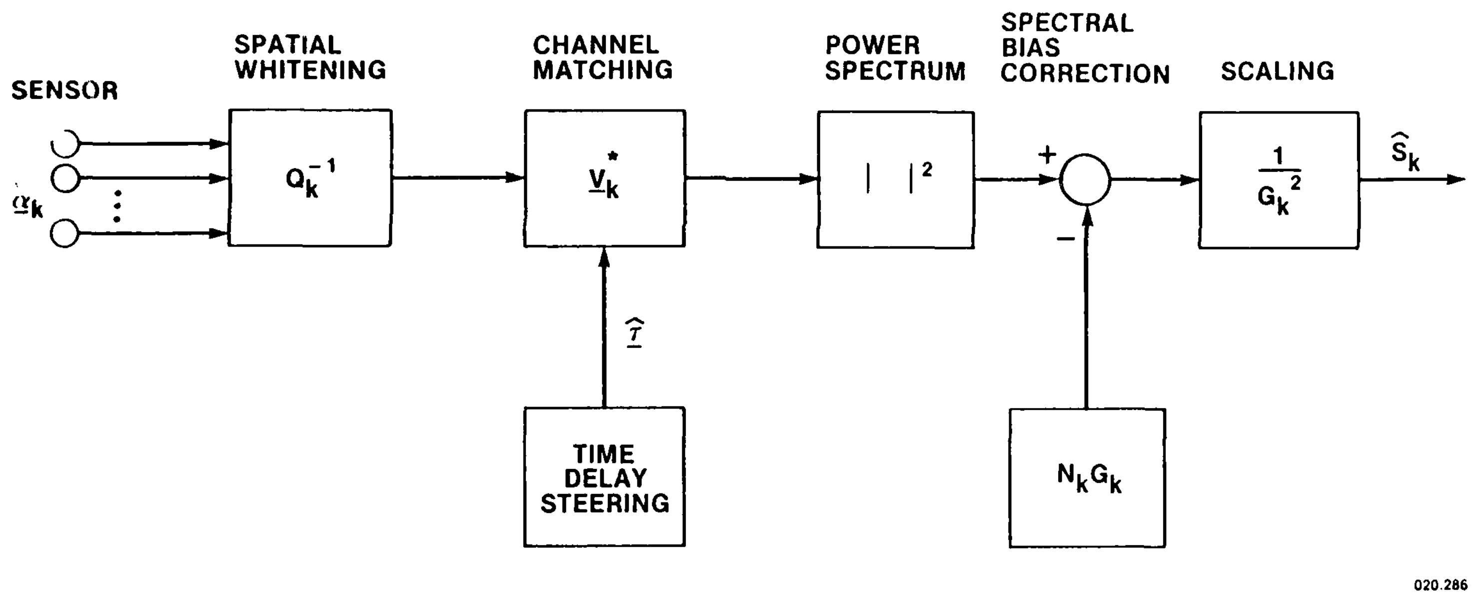

Figure 3-13. Optimum One-Target, Multisensor Spectral Estimator

$$\hat{S}_{k1} = \frac{|\underline{V}_{k1}^{*} \tilde{Q}_{k1}^{-1} \underline{\alpha}_k|^2}{G_{k1}^2} - \frac{\hat{S}_{k2} + N_k}{G_{k1}} . \quad (3.5.2\text{-}12a)$$

$$\hat{S}_{k2} = \frac{|\underline{V}_{k2}^{*} \tilde{Q}_{k2}^{-1} \underline{\alpha}_k|^2}{G_{k2}^2} - \frac{\hat{S}_{k1} + N_k}{G_{k2}} . \quad (3.5.2\text{-}12b)$$

Recall that

$$\tilde{Q}_{k1}^{-1} = (\hat{S}_{k2} + N_k) \, (S_{k2} \, \underline{V}_{k2} \, \underline{V}_{k2}^{*} + N_k \, Q_k)^{-1}$$

$$= (1 + \hat{S}_{k2}/N_k) \left[Q_k^{-1} - \frac{\hat{S}_{k2}/N_k}{1 + M \, \hat{S}_{k2}/N_k} Q_k^{-1} \, \underline{V}_{k2} \, \underline{V}_{k2}^{*} \, Q_k^{-1} \right] . \quad (3.5.3\text{-}13a)$$

Similarly,

$$\tilde{Q}_{k2}^{-1} = (1 + \hat{S}_{k1}/N_k) \left[Q_k^{-1} - \frac{\hat{S}_{k1}/N_k}{1 + M \, \hat{S}_{k1}/N_k} Q_k^{-1} \, \underline{V}_{k1} \, \underline{V}_{k1}^{*} \, Q_k^{-1} \right] . \quad (3.5.3\text{-}13b)$$

Also

$$G_{k1} = \underline{V}_{k1}^{*} \, \tilde{Q}_{k1}^{-1} \, \underline{V}_{k1}$$

$$= (1 + \hat{S}_{k2}/N_k) \left[\underline{V}_{k1}^{*} \, Q_k^{-1} \, \underline{V}_{k1} - \frac{\hat{S}_{k2}/N_k}{1 + M \, \hat{S}_{k2}/N_k} (\underline{V}_{k1}^{*} \, Q_k^{-1} \, \underline{V}_{k1})^2 \right]$$

(3.5.3-13c)

and

$$G_{k2} = (1 + \hat{S}_{k1}/N_k) \left[\underline{V}_{k2}^{*} Q_k^{-1} \underline{V}_{k2} - \frac{\hat{S}_{k1}/N_k}{1 + M \hat{S}_{k1}/N_k} (\underline{V}_{k2}^{*} Q_k^{-1} \underline{V}_{k2})^2 \right]. \tag{3.5.3-13d}$$

For simplicity assume $Q_k = I$, then $\underline{V}_{k1}^{*} Q_k^{-1} \underline{V}_{k1} = V_{k2}^{*} Q_k^{-1} \underline{V}_{k2} = M$. Now the following relations can be obtained:

$$G_{k1} = \frac{M (1 + \hat{S}_{k2}/N_k)}{1 + M \hat{S}_{k2}/N_k} \tag{3.5.3-14a}$$

$$G_{k2} = \frac{M (1 + \hat{S}_{k1}/N_k)}{1 + M \hat{S}_{k1}/N_k} \tag{3.5.3-14b}$$

$$|\underline{V}_{k1}^{*} \tilde{Q}_{k1}^{-1} \underline{\alpha}_k|^2 = (1 + \hat{S}_{k2}/N_k)^2 \left| \underline{V}_{k1}^{*} \underline{\alpha}_k - \frac{\hat{S}_{k2}/N_k}{1 + M \hat{S}_{k2}/N_k} (\underline{V}_{k1}^{*} \underline{V}_{k2}) (\underline{V}_{k2}^{*} \underline{\alpha}_k) \right|^2 \tag{3.5.3-14c}$$

$$|\underline{V}_{k2}^{*} \tilde{Q}_{k2}^{-1} \underline{\alpha}_k|^2 = (1 + \hat{S}_{k1}/N_k)^2 \left| \underline{V}_{k2}^{*} \underline{\alpha}_k - \frac{\hat{S}_{k1}/N_k}{1 + M \hat{S}_{k1}/N_k} (\underline{V}_{k1}^{*} \underline{V}_{k2}) (\underline{V}_{k1}^{*} \underline{\alpha}_k) \right|^2 . \tag{3.5.3-14d}$$

Substituting the above into Equation (3.5.3-12a-b) and simplifying yields

$$\hat{S}_{k1} = A_{k2}\left[A_{k2}\ |\underline{V}_{k1}^{*}\ \underline{\alpha}_k - B_{k2}\ \underline{V}_{k2}^{*}\ \underline{\alpha}_k|^2 - N_k\right] \tag{3.5.3-15a}$$

$$\hat{S}_{k2} = A_{k1}\left[A_{k1}\ |\underline{V}_{k2}^{*}\ \underline{\alpha}_k - B_{k1}\ \underline{V}_{k1}^{*}\ \underline{\alpha}_k|^2 - N_k\right] \tag{3.5.3-15b}$$

where A_{ki} and B_{ki} are defined by

$$A_{ki} = \frac{1 + M\ \hat{S}_{ki}/N_k}{M}\ ; \qquad i = 1,\ 2 \tag{3.5.3-15c}$$

and

$$B_{ki} = \frac{\hat{S}_{ki}/N_k}{1 + M\ \hat{S}_{ki}/N_k}\ \underline{V}_{k1}^{*}\ \underline{V}_{k2}; \qquad i = 1,\ 2\ . \tag{3.5.3-15d}$$

The optimum two-target M sensor spectral estimator is shown in Figure 3-14.

We have already shown that the optimum spectral estimator is an unbiased estimator. In the remainder of this section we briefly discuss the estimator performance bound. There are two cases of particular interest: (1) power spectral estimation with known time delay, and (2) joint time delay and spectral estimation.

3.5.3.3 Case 1: Power Spectral Estimation with Known Time Delay. Consider the two-target M sensor problem. The optimum power spectral estimator is given in Figure 3-14. The spectral CRLB can be derived as follows.

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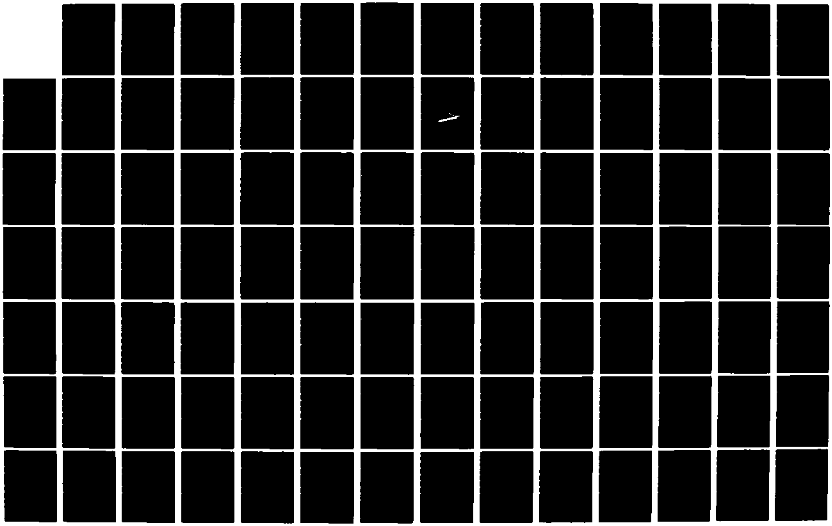
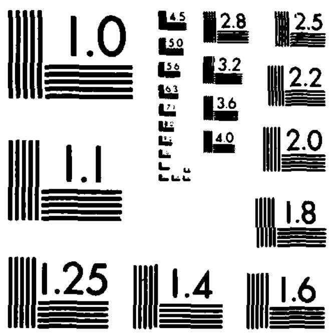
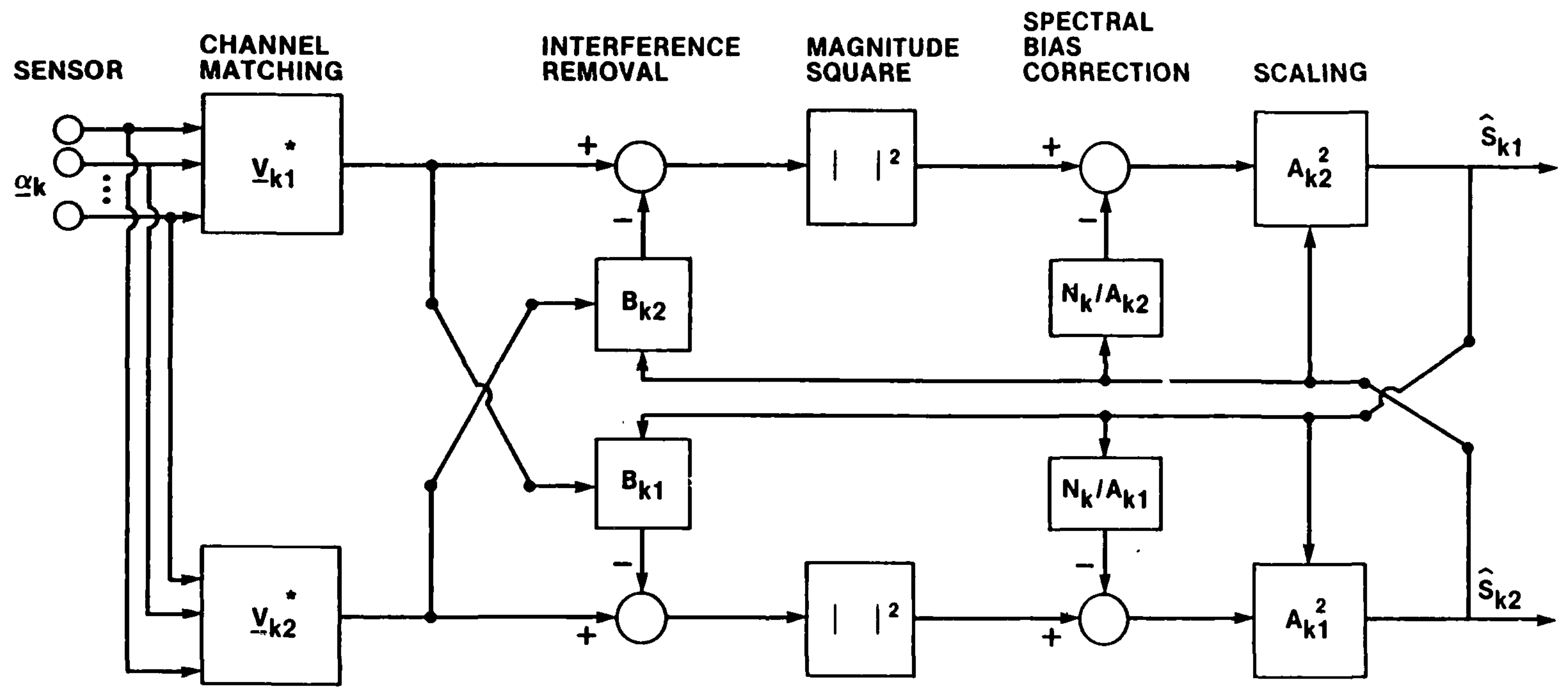




MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A



$$A_{ki} = \frac{1 + M\,\hat{S}_{ki}/N_k}{M}\,;\quad B_{ki} = \frac{\hat{S}_{ki}/N_k}{1 + M\,\hat{S}_{ki}/N_k}$$

FOR i = 1 AND 2

Figure 3-14. Optimum Two-Target Power Spectral Estimator

For each frequency ω_k, we have the relation

$$\mathrm{VAR}(\hat{S}_{kj}) = [F^{-1}]_{jj} \; ; \qquad j = 1, 2 \tag{3.5.3-16}$$

where F is the 2x2 Fisher information matrix whose $j\ell$ element is given by

$$\begin{aligned} F_{j\ell} &= -E\left[\frac{\partial^2 \Lambda(\underline{S})}{\partial S_{kj} \, \partial S_{k\ell}} \right] \\ &= \mathrm{tr}\left(R_k^{-1} \frac{\partial R_k}{\partial S_{kj}} R_k^{-1} \frac{\partial R_k}{\partial S_{k\ell}} \right) \\ &= -\mathrm{tr}\left(\frac{\partial R_k^{-1}}{\partial S_{kj}} \frac{\partial R_k}{\partial S_{k\ell}} \right) \end{aligned} \tag{3.5.3-17}$$

where Equations (3.5.3-2) and (3.4-4) have been used. Using Equations (3.5.3-3a) and (3.5.3-3b), Equation (3.5.3-17) reduces to

$$\begin{aligned} F_{j\ell} &= \mathrm{tr}\left(\frac{\tilde{Q}_{kj}^{-1} \, \underline{V}_{kj} \, \underline{V}_{kj}^{*} \, \tilde{Q}_{kj}^{-1} / \tilde{N}_{kj}^{2}}{(1 + G_{kj} \, S_{kj}/\tilde{N}_{kj})^2} \, \underline{V}_{k\ell} \, \underline{V}_{k\ell}^{*} \right) \\ &= \frac{|\underline{V}_{k\ell}^{*} \, \tilde{Q}_{kj}^{-1} \, \underline{V}_{kj}^{*}|^2 / \tilde{N}_{kj}^{2}}{(1 + G_{kj} \, S_{kj}/\tilde{N}_{kj})} \quad . \end{aligned} \tag{3.5.3-18a}$$

Note for $j = \ell$, Equation (3.5.3-18a) further reduces to

$$F_{jj} = \left[\frac{G_{kj}/\tilde{N}_{kj}}{1 + G_{kj} \; S_{kj}/\tilde{N}_{kj}} \right]^2 . \qquad (3.5.3\text{-}18b)$$

Thus, the spectral CRLB is

$$\mathrm{VAR}(\hat{S}_{kj}) = \frac{1}{(1 - M_{12}^2)} \; \frac{1}{F_{jj}}$$

$$= \frac{1}{(1 - M_{12}^2)} \left(S_{kj} + \frac{\tilde{N}_{kj}}{G_{kj}} \right)^2 ; \qquad j = 1, 2 \qquad (3.5.3\text{-}19a)$$

where

$$M_{12} = \frac{F_{12}}{(F_{11} \; F_{22})^{1/2}}$$

$$= \frac{\left| \underline{V}_{k2}^{*} \; \tilde{Q}_{k1}^{-1} \; \underline{V}_{k1}^{*} \right|^2}{(G_{k1} \; G_{k2})} \qquad (3.5.3\text{-}19b)$$

is the spectral coefficient of mutual dependence. Note that for a single target case, the spectral CRLB is

$$\mathrm{VAR}(\hat{S}_k) = -\left\{ E\left[\frac{\partial^2 \Lambda(\underline{S})}{\partial S_k^2}\right]\right\}^{-1}$$

$$= \left\{ \mathrm{tr}\left(\frac{\partial R_k^{-1}}{\partial S_k} \; \frac{\partial R_k}{\partial S_k}\right)\right\}^{-1}$$

$$= \left[S_k + \frac{N_k}{G_k}\right]^2 \qquad (3.5.3\text{-}20)$$

where $G_k = \underline{V}_k^* \, Q_k^{-1} \, \underline{V}_k$ is known as the array gain.

3.5.3.4 Case 2: Joint Time Delay and Spectral Estimation. For simplicity, we assume a single target and two sensors. Therefore, the unknowns are τ, the time delay, and S_k, the spectral level at frequency ω_k. Note that because the spectral likelihood equation is independent for each frequency ω_k, we only need to consider the joint estimate between τ and S_k. Now let $\theta_1 = \tau$, and $\theta_2 = S_k$, then the joint time delay spectral CRLB evaluated at the true parameter values is

$$\mathrm{VAR}(\hat{\theta}_i) \geq (F^{-1})_{ii} , \qquad i = 1, 2 \qquad (3.5.3\text{-}21a)$$

where the ij element of the Fisher information matrix is given by

$$F_{ij} = -E\left(\frac{\partial^2 \Lambda(\underline{S},\tau)}{\partial \theta_i \; \partial \theta_j}\right). \qquad (3.5.3\text{-}21b)$$

Now

$$F_{11} = -E\left(\frac{\partial^2 \Lambda(\underline{S},\tau)}{\partial \tau^2}\right)$$

$$= -\sum_{k=1}^{B} \operatorname{tr}\left(\frac{\partial R_k^{-1}}{\partial \tau} \frac{\partial R_k}{\partial \tau}\right)$$

$$= 2\sum_{k=1}^{B} \omega_k^2 |h_k|^2 S_k \qquad (3.5.3\text{-}21c)$$

where Equation (3.5.1-15c) has been used. Also $|h_k|^2 = (S_k/N_k^2)/(1 + 2\, S_k/N_k)$.

From Equation (3.5.3-18b) we obtain

$$F_{22} = -E\left(\frac{\partial^2 \Lambda(\underline{S},\tau)}{\partial S_k^2}\right)$$

$$= \left(\frac{G_k/N_k}{1 + G_k\, S_k/N_k}\right)^2 . \qquad (3.5.3\text{-}21d)$$

Finally, the cross term is

$$F_{12} = -E\left(\frac{\partial^2 \Lambda(\underline{S},\tau)}{\partial\tau\ \partial S_k}\right)$$

$$= -\mathrm{tr}\left(\frac{\partial R_k^{-1}}{\partial\tau}\ \frac{\partial R_k}{\partial S_k}\right)$$

$$= j\omega_k |h_k|^2\ \mathrm{tr}(V_k\ \Phi_1\ V_k^*\ V_k\ 1_M\ V_k^*) \qquad (3.5.3\text{-}21e)$$

where relations in Equations (3.4-2) and (3.4.-5b) have been used. But from Equation (3.5.1-13e), we obtain

$$\mathrm{tr}(V_k\ \Phi_1\ V_k^*\ V_k\ 1_M\ V_k^*) = \mathrm{tr}\left\{\begin{bmatrix}1 & 0\\ 0 & e^{-j\omega\tau}\end{bmatrix}\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & e^{-j\omega\tau}\end{bmatrix}\right.$$

$$\left.\begin{bmatrix}1 & 0\\ 0 & e^{j\omega\tau}\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & e^{-j\omega\tau}\end{bmatrix}\right\}$$

$$= \mathrm{tr}\begin{bmatrix}-1 & -e^{-j\omega\tau}\\ e^{j\omega\tau} & 1\end{bmatrix}$$

$$= 0\ . \qquad (3.5.3\text{-}21f)$$

Thus, $F_{12} = 0$, indicating that spectral estimates and time delay estimate are uncorrelated. Hence, using Equation (3.5.3-21a) the CRLB is given by

$$\mathrm{VAR}(\hat{\tau}) = \left(2 \sum_{k=1}^{B} \omega_k^2 \, |h_k|^2 \, S_k\right)^{-1} \tag{3.5.3-22a}$$

and

$$\mathrm{VAR}(\hat{S}_k) = \left(S_k + \frac{N_k}{G_k}\right)^2 \tag{3.5.3-22b}$$

which is identical to the time delay estimate with a known spectrum and to the spectral estimate with a known time delay, respectively. Thus, we conclude that joint time delay spectral estimation does not degrade the time delay estimates nor the spectral estimates.

It is interesting and revealing to show the explicit dependence of the time delay spectral performance on the observation time T. Let $S(\omega)$ and $N(\omega)$ be the true continuous signal and noise power spectra. Now recall from Equations (2-13d) and (3.5.1-13d) that $S_k = E(\beta_k \beta_k^*)$ and $|h_k|^2 = S_k/N_k^2/(1 + 2\, S_k/N_k)$, where β_k is the Fourier coefficient of the signal waveform from T seconds of observation. Also recall the relation that for sufficiently large T, we have $TS_k = S(\omega_k)$, $TN_k = N(\omega_k)$, and $|h_k|^2 = T|h(\omega_k)|^2$. Consequently, Equations (3.5.1-22a) and (3.5.1-22b) can be manipulated as follows to yield

$$\mathrm{VAR}(\hat{\tau}) = \left[\frac{T}{\pi} \sum_{k=1}^{B} \omega_k^2 \, |h(\omega_k)|^2 \, S(\omega_k) \quad \frac{2\pi}{T}\right]^{-1}$$

$$= \frac{\pi}{T} \left[\int_0^B \omega^2 \left(\frac{S^2(\omega)/N^2(\omega)}{1 + 2\, S(\omega)/N(\omega)} \right) d\omega \right]^{-1} \tag{3.5.3-23a}$$

$$\mathrm{VAR}(\hat{S}_k) = \frac{1}{T^2}\left[(TS_k) + \frac{(TN_k)}{G_k}\right]^2$$

$$= \frac{1}{T^2}\left[S(\omega_k) + \frac{N(\omega_k)}{G(\omega_k)}\right]^2 \qquad (3.5.3\text{-}23b)$$

where $S(\omega_k)$ denotes $S(\omega)$ evaluated at frequency $\omega = 2\pi k/T$.

Thus, Equations (3.5.3-23a) and (3.5.3-23b) indicate that while the time delay variance is inversely proportional to T, the spectral variance is inversely portional to the square of T. In other words, the spectral variance is more effectively reduced by increasing the observation time than the time delay variance.

CHAPTER 4
SUBOPTIMUM REALIZATION OF MULTISENSOR, MULTITARGET TIME DELAY PROCESSOR

4.1 INTRODUCTION

In Chapter 3 we derived the optimum (MLE) multisensor, multitarget time delay estimator. The result yields a highly coupled multi-channel processor. For practical applications, it is desired to seek suboptimum realizations which can substantially simplify the required implementation. In this section we examine the suboptimum processor based on a weak signal in noise assumption. This is the case of considerable interest since in passive signal processing, a weak signal in noise represents the usual environment at which a signal processor must operate.

4.2 WEAK SIGNAL IN NOISE SUBOPTIMUM PROCESSOR REALIZATION

Assuming that

$$S(\omega)_j/N(\omega) \ll 1 \text{ for } j = 1, 2, \ldots, J \tag{4.2-1a}$$

and using Equations (3.5.1-17c-g), Equation (3.5.1-12) becomes

$$|h_j(\omega)|^2 = \frac{S_j(\omega)/\tilde{N}_j^2(\omega)}{1 + G_j(\omega)\ S_j(\omega)/\tilde{N}_j(\omega)} \approx \frac{S_j(\omega)/N_j^2(\omega)}{1 + M\ S_j(\omega)/N(\omega)} \tag{4.2-1b}$$

$$\tilde{a}_j(\omega) = \tilde{N}_j(\omega)|\tilde{h}_j(\omega)|^2 \approx \frac{S_j(\omega)/\tilde{N}_j(\omega)}{1 + M\ S_j(\omega)/N(\omega)} \tag{4.2-1c}$$

$$\tilde{Q}_j^{-1}(\omega) = \tilde{N}_j(\omega) \left[\sum_{\substack{i=1 \\ i\neq j}}^{J} S_i(\omega)\ P_i(\omega) + N(\omega)\ Q(\omega) \right]^{-1} \tag{4.2-1d}$$

$$\approx Q^{-1}(\omega)\ \tilde{N}_j(\omega)/N(\omega) \tag{4.2-1e}$$

$$\tilde{N}_j(\omega) = \sum_{\substack{i=1 \\ i\neq j}}^{J} S_i(\omega) + N(\omega) \tag{4.2-1f}$$

$$P_j(\omega) = V_j(\omega)\ 1_M\ V_j^*(\omega) \tag{4.2-1g}$$

$$\tilde{b}_i^j(\omega) \approx \left[\frac{S_j(\omega)/\tilde{N}_j(\omega)}{1 + M\ S_j(\omega)/N(\omega)}\right] \mathrm{tr}\left[\tilde{Q}_j^{-1}\ V_j(\omega)\ \Phi_i\ V_j^*(\omega)\right] \tag{4.2-1h}$$

and the simplified likelihood Equation (3.5.1-12) of estimating time delay between sensors i and i + 1 due to target j becomes:

$$Z_{ij}(\underline{\tau}) = \frac{\partial\Lambda(\underline{\tau})}{\partial\tau_{ij}}$$

$$= \int_0^\infty \Big\{ j\omega\ \ |h_j(\omega)|^2\ \underline{\alpha}^*(\omega)\ Q^{-1}(\omega)\ V_j(\omega)[\Phi_i - \tilde{b}_i^j(\omega)\ 1_M]$$

$$V_j^*(\omega)\ Q^{-1}(\omega)\ \underline{\alpha}(\omega) - T\ \tilde{b}_i^j(\omega) \Big\} \frac{d\omega}{2\pi} \tag{4.2-2}$$

Note that Equation (4.2-2) results in a substantial simplification of Equation (3.5.1-12). For the two-target and two-sensor case studied in Section 3.. , the esulting two likelihood equations (3.5.1-17a) and (3.5.1-17b) become:

$$\frac{\partial\Lambda(\tau_1,\ \tau_2)}{\partial\tau_1} = \int_0^\infty j\omega \left\{ |h_1|^2\ \underline{\alpha}^*\ Q^{-1}\ V_1(\Phi_1 - b_1\ 1_M)\ V_1^*\ Q^{-1}\ \underline{\alpha} - T\ b_1 \right\} \frac{d\omega}{2\pi}$$

$$= 0 \tag{4.2-3a}$$

$$\frac{\partial\Lambda(\tau_1,\ \tau_2)}{\partial\tau_2} = \int_0^\infty j\omega \left\{ |h_2|^2\ \underline{\alpha}^*\ Q^{-1}\ V_2(\Phi_1 - b_2\ 1_M)\ V_2^*\ Q^{-1}\ \underline{\alpha} - T\ b_2 \right\} \frac{d\omega}{2\pi}$$

$$= 0 \tag{4.2-3b}$$

where

$$|h_1|^2 = \frac{S_1/(S_2 + N)^2}{1 + 2\ S_1/N} \tag{4.2-3c}$$

$$|h_2|^2 = \frac{S_2/(S_1 + N)^2}{1 + 2\ S_2/N} \tag{4.2-3d}$$

$$Q^{-1} = I \tag{4.2-3e}$$

$$\Phi_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{4.2-3f}$$

$$b_1 = -\left(\frac{S_1/N}{1 + 2\ S_1/N}\right)\left(\frac{S_2/N}{1 + 2\ S_2/N}\right)\left[e^{j\omega(\tau_1-\tau_2)} - e^{-j\omega(\tau_1-\tau_2)}\right] \tag{4.2-3g}$$

$$b_2 = -\left(\frac{S_1/N}{1 + 2\ S_1/N}\right)\left(\frac{S_2/N}{1 + 2\ S_2/N}\right)\left[e^{j\omega(\tau_2-\tau_1)} - e^{-j\omega(\tau_2-\tau_1)}\right]. \tag{4.2-3h}$$

Using Equations (4.2-3c) through (4.2-3h), Equations (4.2-3a) and (4.2-3b) can be further simplified to

$$\frac{\partial\Lambda(\tau_1,\ \tau_2)}{\partial\tau_1} = \frac{T}{2\pi}\int_{-\infty}^{\infty} j\omega\left\{|h_1|^2\ \tilde{G}_{x_1x_2} + \frac{S_1S_2/N^2}{(1+2\ S_1/N)\ (1+2\ S_2/N)}\left(1+\frac{1}{T}\ |h_1\ \underline{v}_1^*\ Q^{-1}\ \underline{\alpha}|^2\right) e^{-j\omega\tau_2}\right\} e^{j\omega\tau_1}\ d\omega$$

$$= 0 \qquad (4.2\text{-}4a)$$

$$\frac{\partial\Lambda(\tau_1,\ \tau_2)}{\partial\tau_2} = \frac{T}{2\pi}\int_{-\infty}^{\infty} j\omega\left\{|h_2|^2\ \tilde{G}_{x_1x_2} + \frac{S_1\ S_2/N^2}{(1+2\ S_1/N)\ (1+2\ S_2/N)}\left(1+\frac{1}{T}\ |h_2\ \underline{v}_2^*\ Q^{-1}\underline{\alpha}|^2\right) e^{-j\omega\tau_1}\right\} e^{j\omega\tau_2}\ d\omega$$

$$= 0 \qquad (4.2\text{-}4b)$$

where $\underline{x} = (x_1\ x_2)^T = Q^{-1}\ \underline{\alpha}$, $\hat{G}_{x_1x_2}$ is the estimated cross power spectrum, and $\underline{v}_1$, $\underline{v}_2$ are the time delay steering vectors. A block diagram of this processor is shown in Figure 4-1. Note that given two sensors, a single channel GCC under a multiple target environment is known to be biased. The coupling shown in Figure 4-1 provides the required bias correction. Note that the bias correction term is a function of SNR and INR. When the power spectra are not known, the quantities SNR and INR must be substituted by their estimated values. Thus, the optimum power spectral estimator discussed in Section 3.5.3 is applicable. Finally, if the signal spectrum and interference spectrum are separable (i.e., no overlapped region), the two channels become uncoupled.

4.3 A SINGLE TARGET ASSUMPTION SUBOPTIMUM PROCESSOR

For completeness we now include a study on the suboptimum multitarget processor using a single target formulation. The resulting processor performance has been studied widely in the context of the localization variables (i.e., range and bearing) (References 7, 10, 11, 12, 23, 24). However, a direct study on the time delay variable has not been seen in the open literature.

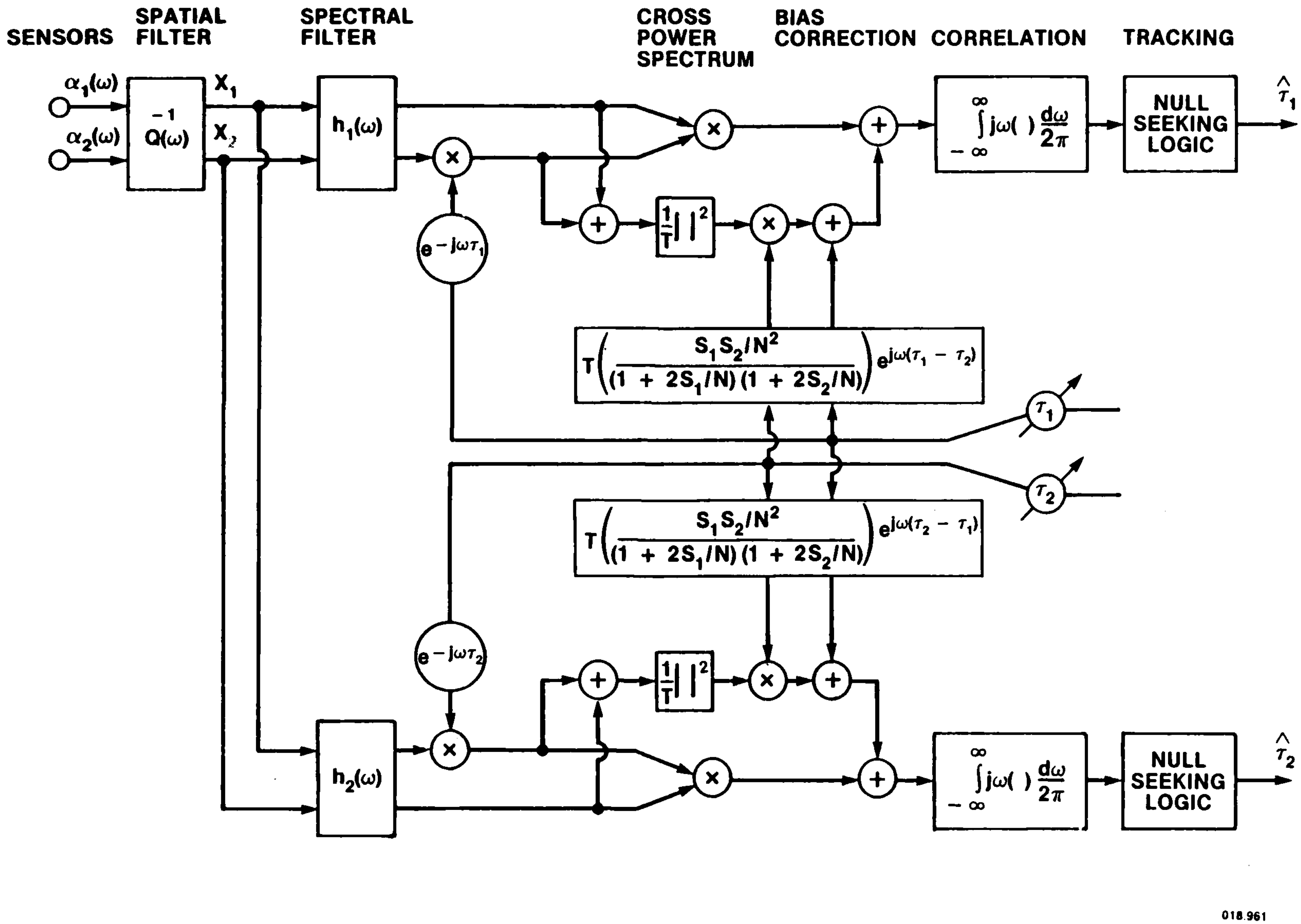


Figure 4-1. Suboptimum Two-Sensor, Two-Target Time Delay Processor

A single target processor can be obtained from the multitarget processor by setting the interference power spectra to zero. For the two-sensor case, the resulting processor reduces to the GCC processor discussed in Section 3.5.1. The time delay is obtained from the GCC by locating the peak of the GCC function (assuming that SNR is sufficiently high so that a dominant peak can be detected). In the presence of interference the resulting estimates are known to be biased. In addition, they affect the time delay variance performance. Here we shall quantify the performance in more detail.

Consider a general two-sensor J target problem. Let the true time delay to target j be τ_j for j = 1, 2, ..., J. Then the frequency domain representation from T seconds of observation time can be written as

$$\alpha_{1k} = \underline{v}_{1k}\, \beta_k + n_{1k} \tag{4.3-1a}$$

$$\alpha_{2k} = \underline{v}_{2k}\, \beta_k + n_{2k}\ ; \qquad k = 1, 2, \ldots, B \tag{4.3-1b}$$

where the complex vectors $\underline{v}_{1k}$, $\underline{v}_{2k}$ and β_k are defined by

$$\underline{v}_{ik} = \left(e^{j\omega_k D_{i1}}\ e^{j\omega_k D_{i2}} \ldots e^{j\omega_k D_{iJ}} \right) \ ; \qquad i = 1, 2 \tag{4.3-2a}$$

and

$$\underline{\beta}_k = \left(\beta_{k1}\ \beta_{k2} \cdots \beta_{kJ} \right)^T \tag{4.3-2b}$$

where D_{ij} is the propagation time delay from target j to sensor i and β_{kj} is the Fourier component of the signal spectrum of target j. The peak of the GCC is obtained by locating the null of the function (see Equation 3.5.1-14c)

$$f(\tau) = \sum_{k=-B}^{B} j\omega_k \; |h_k|^2 \; \alpha_{1k} \; \alpha_{2k}^* \; e^{j\omega_k \tau} \tag{4.3-3a}$$

where $|h_k|^2$, the spectral shaping filter, is given by

$$|h_k|^2 = \frac{S_{k1}/N_k^2}{1 + 2\; S_{k1}/N_k} \quad . \tag{4.3-3b}$$

Note that without loss of generality we have let $j = 1$ be the target of interest and let the remaining $J - 1$ targets be interferences. An alternate selection of the frequency shaping filter is the multitarget spectral shaping filter (see Equation (3.5.1-4b)):

$$|\tilde{h}_k|^2 = \frac{S_{k1}/\tilde{N}_{k1}^2}{1 + G_{k1}\; S_{k1}/\tilde{N}_{k1}} \tag{4.3-4a}$$

where from Equations (3.5.1-2c) and (3.5.1-3)

$$\tilde{N}_{k1} = \sum_{j=2}^{J} S_{kj} + N \tag{4.3-4b}$$

and

$$\begin{aligned} G_{k1} &= \underline{V}_{k1}^* \; \tilde{Q}_{k1}^{-1} \; \underline{V}_{k1} \\ &= \underline{V}_{k1}^* \left[\tilde{N}_{k1} \left(\sum_{j=2}^{J} S_{kj} \; P_{kj} + N_k \; Q_k \right)^{-1} \right] \underline{V}_{k1} \end{aligned}$$

$$\simeq \frac{\tilde{N}_{k1}}{N_k} \underline{V}^*_{k1} Q_k^{-1} \underline{V}_{k1}$$

$$\simeq 2 \frac{\tilde{N}_{k1}}{N_k} \tag{4.3-4c}$$

for the two sensors with $Q_k = I$.

Therefore, a simpler form of Equation (4.3-4a) is

$$|\tilde{h}_k|^2 = \frac{S_{k1}/\tilde{N}^2_{k1}}{1 + 2\, S_{k1}/N_k} \quad . \tag{4.3-4d}$$

For example, for the two-target case, Equation (4.3-4d) reduces to

$$|\tilde{h}_k|^2 = \frac{S_{k1}/(S_{k2} + N_k)^2}{1 + 2\, S_{k1}/N_k} \quad . \tag{4.3-4e}$$

The basic derivation of the bias and variance is shown in Appendix H (Equations (H-8e) and (H-17b)). The expressions for the bias and the variance for the two-target case with identically flat signal, interference, and noise power spectra are given by Equations (H-12) and (H-21a) as

$$b_1 = \frac{1}{1 + S/I} \; (\tau_2 - \tau_1) \tag{4.3-5a}$$

and

$$\mathrm{VAR}(\hat{\tau}_1) = \frac{2\pi}{T}\left[\frac{1 - (a_1^2\,\rho(2\Delta_1) + a_2^2\,\rho(2\Delta_2) + 2a_1\,a_2\,\rho(\Delta_1 + \Delta_2))}{2(a_1\,\rho(\Delta_1) + a_2\,\rho(\Delta_2))^2\,R(0)}\right] \tag{4.3-5b}$$

where S/I is the signal-to-interference ratio and

$$a_1 = \frac{S/N}{1 + S/N + I/N} \tag{4.3-6a}$$

$$a_2 = \frac{I/N}{1 + S/N + I/N} \tag{4.3-6b}$$

$$\Delta_1 = \hat{\tau}_1 - \tau_1 \tag{4.3-6c}$$

$$\Delta_2 = \hat{\tau}_1 - \tau_2 \tag{4.3-6d}$$

$$R(\tau) = \int_{\omega_1}^{\omega_2} \omega^2 \cos \omega\tau \, d\omega \tag{4.3-6e}$$

$$\rho(\tau) = R(\tau)/R(0) \ . \tag{4.3-6f}$$

Note that in the limit as $\Delta_1 \to 0$, $\Delta_2 \to \infty$, we have $\rho(\Delta_2) \to 0$, $\rho(\Delta_1) \to 1$ and the steady state variance is

$$\mathrm{VAR}_{\infty}(\hat{\tau}_1) = \lim_{\substack{\Delta_1 \to 0 \\ \Delta_2 \to \infty}} \mathrm{VAR}(\hat{\tau}_1)$$

$$= \frac{2\pi}{T} \left(\frac{1 - a_1^2}{2\, a_1^2} \right) \frac{1}{R(0)} \quad . \tag{4.3-7}$$

Therefore, one can define a normalized variance as

$$\frac{\mathrm{VAR}(\hat{\tau}_1)}{\mathrm{VAR}_{\infty}(\hat{\tau}_1)} = \left(\frac{a_1^2}{1 - a_1^2} \right) \left[\frac{1 - (a_1^2\, \rho(\Delta_1) + 2a_1\, a_2\, \rho(\Delta_1 + \Delta_2) + a_2^2\, \rho(\Delta_2))}{2(a_1\, \rho(\Delta_1) + a_2\, \rho(\Delta_2))^2} \right]. \tag{4.3-8}$$

Note that Equation (4.3-7) does not reduce to the single target case because of the presence of spectral interference. This is certainly true for the case of two omni-directional sensor arrays since interference power is not spatially attenuated as a function of time delay separation. However, for sensor arrays with large base length separation, this condition is again satisfied since a small spatial separation could produce a large time delay separation and insignificant spatial attenuation.

4.4 GCC PERFORMANCE IN THE PRESENCE OF INTERFERENCE

Performance of the GCC in the presence of interference is compared to the optimum processor in the remainder of this section. Figure 4-2 shows the interference of the expected value of the GCC due to signal only by the GCC from a second target with an identical spectrum shape but a 3 dB smaller signal power. It can be seen that (1) the two GCCs are merged to one (i.e., it fails to resolve the target from the interference), and (2) the peak of the combined GCC is biased. Figure 4-3 shows the same GCC in the form of a 3-D interference pattern as a function of target-interference time delay separation.

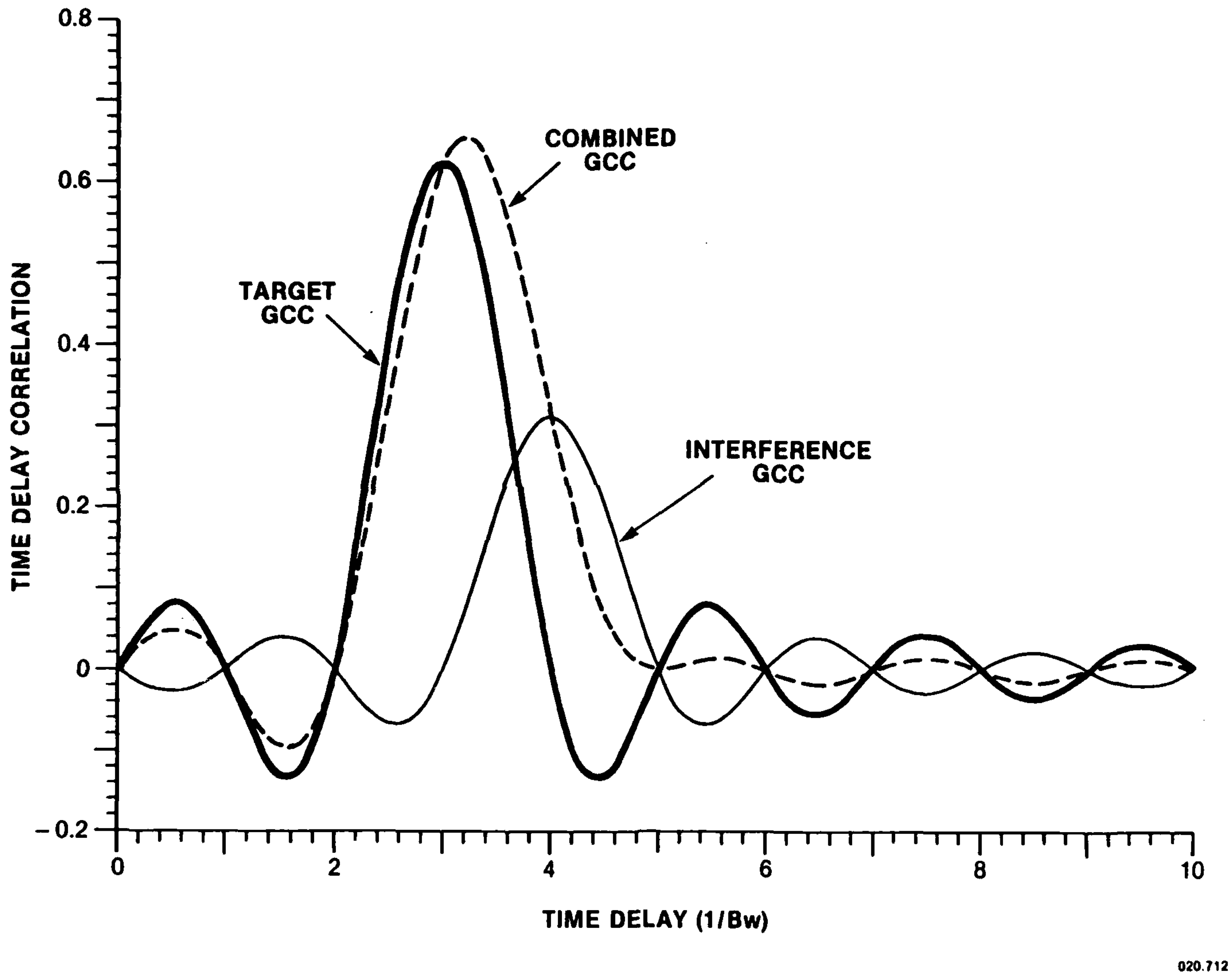


Figure 4-2. Multitarget Time Delay Interference (SNR = 10 dB, INR = 7 dB)

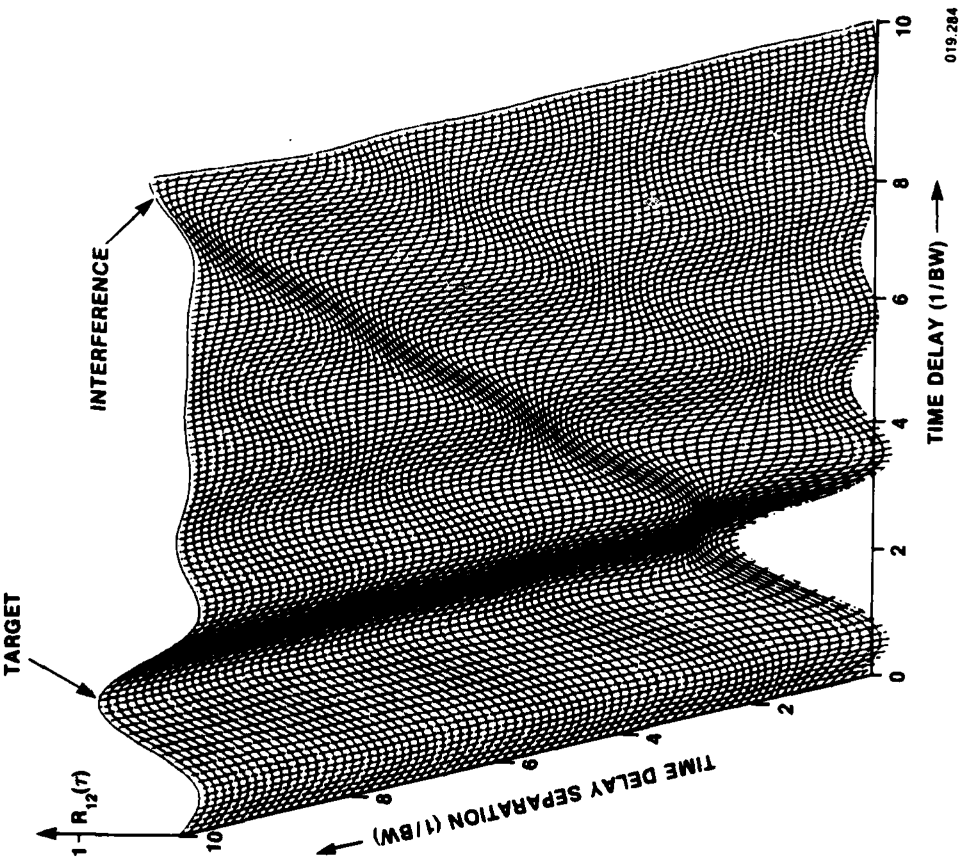


Figure 4-3. Multitarget Time Delay Interference as a Function of Time Delay Separation (SNR = 10 dB, INR = 7 dB)

(Note the combined GCC in Figure 4-2 corresponds to the curve with unity separation in Figure 4-3.) Since the MLE is asymptotically an unbiased estimator, the optimum processor has no bias for sufficient integration time. Therefore, it resolves the multitarget ambiguity. This is the primary advantage of the optimum multitarget processor. Figure 4-4 shows the bias characteristics of the GCC. Note that when the time delay separation is small, the bias is proportional to the separation, indicated by Equation (4.3-5a). Figure 4-5 presents the resulting GCC variance. The GCC variance about the estimated mean is normalized by the corresponding variance with no interference (Equation (3.5.1-16b)). Also shown in Figure 4-5 is the CRLB of the optimum two-target processor. This optimum processor not only resolves the bias but also has a smaller steady state variance. Note that even with no time delay interference, the variance of the GCC does not approach the bound because of the presence of interference in the frequency domain. Also note that with no time delay separation between target and interference, the CRLB is singular, reflecting the inherent inappropriateness of a two-target formulation to a one-target estimation problem.

Also shown in Figure 4-5 is the total Root Mean Square (RMS) error of the conventional GCC processor in the presence of interference. Note that the CRLB of the optimum two-target processor results in a significantly reduced total RMS error. Finally, we remark that identical signal and interference power spectra are the worst conditions. For different signal and interference spectra, the optimum processor is well behaved even at zero time delay separation (see Figure 3-7).

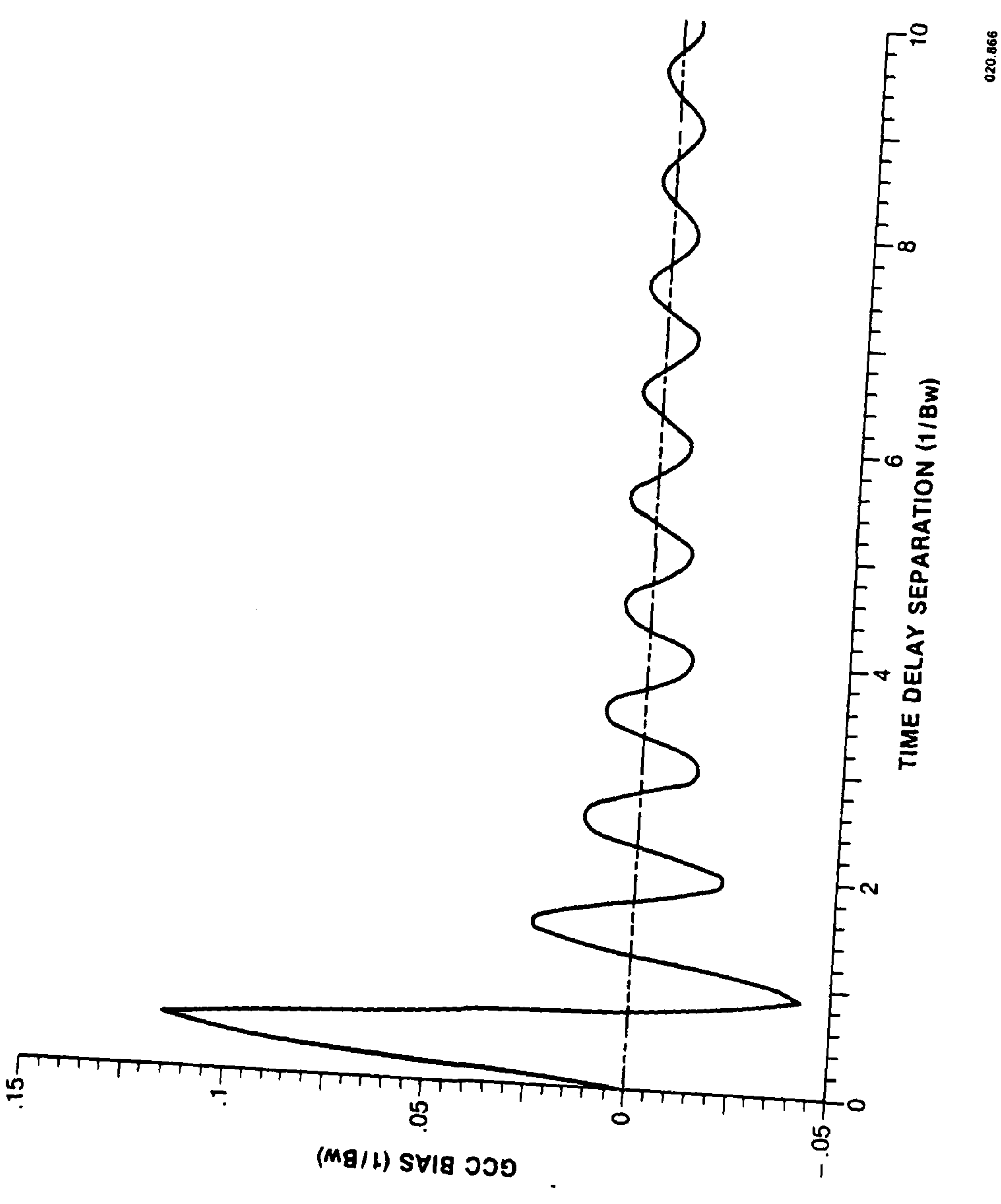


Figure 4-4. GCC Bias in the Presence of Interference (SNR = 10 dB, INR = 7 dB)

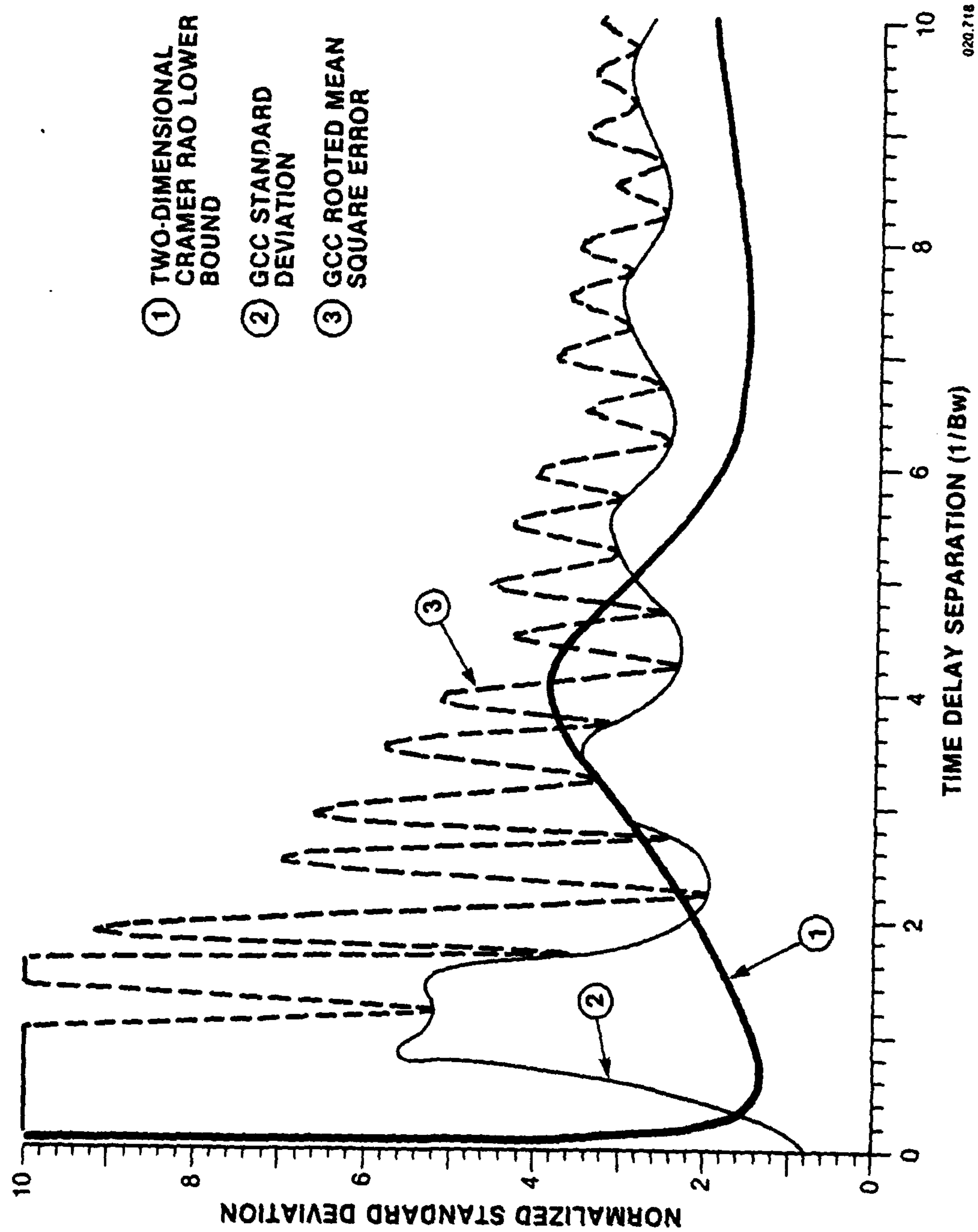


Figure 4-5. Normalized GCC Variance in the Presence of Interference (SNR = 10 dB, INR = 7 dB)

CHAPTER 5
IMPROVED MULTITARGET PARAMETER RESOLUTION

5.1 INTRODUCTION

In Chapter 3 we studied the optimum multitarget multisensor parameter estimator using the MLE procedure. In Chapter 4 we studied a number of suboptimum realizations. We pointed out that the GCC processor can be considered as a suboptimum processor in a multitarget environment. The use of GCC in a multitarget environment results in a poor multitarget parameter resolution; i.e., it yields a single biased estimate when separation between targets is small. The use of an Optimum Multitarget Processor (OMP) can provide a significant improvement in resolution (as shown in Figure 4-5). Unfortunately, the OMP also requires a major modification of many existing systems where GCC processors have already been implemented. Therefore in this chapter we investigate alternate procedures for improved multitarget parameter resolution.

The particular approach to be studied is the post GCC multitarget processor. In this approach, additional multitarget processing capability is provided at the GCC outputs. Thus with this approach, the GCC processor needs no modification. In effect, the existing GCC processor can serve conveniently as a pre-processor of a multitarget estimator.

This chapter is organized as follows. Section 5.2 derives the post GCC multitarget processor. Section 5.3 presents the estimator performance bound. Section 5.4 discusses the simulation procedure and results. Specifically, we compare the performance of the conventional GCC processor, the post GCC multitarget processor, and the optimum multitarget processor.

5.2 POST GCC MULTITARGET PROCESSOR

From Equation (3.5.1-14b), the GCC output can be written as

$$R(\tau) = \sum_{k=-B}^{B} |h_k|^2 \alpha_{1k} \alpha_{2k}^* e^{j\omega_k \tau} \tag{5.2-1}$$

where $|h_k|$, α_{1k} and α_{2k}^* are defined as shown in Appendix H. Now Equation (5.2-1) can be written as

$$R(\tau) = \overline{R}(\tau) + W(\tau) \tag{5.2-2a}$$

where $\overline{R}(\tau)$ denotes the deterministic component and $W(\tau)$ the random component of the noisy GCC ouput. For practical application, Equation (5.2-1) is usually realized via a Fast Fourier Transform (FFT). Thus the GCC output consists of a discrete set of observations. Therefore, let Δt be the sampling time, then the discrete GCC output can be written as

$$R(n\Delta t) = \overline{R}(n\Delta t) + W(n\Delta t) \ ; \qquad n = 0, \pm 1, \pm 2, \ldots, \pm N/2 \ . \tag{5.2-2b}$$

Using Equation (5.2-1), the deterministic and the random components of $R(n\Delta t)$ can be obtained. For example, taking the expected value on both sides of Equation (5.2-1), the deterministic component is

$$\overline{R}(n\Delta t) = \sum_{k=-B}^{B} |h_k|^2 \overline{\alpha_{1k} \alpha_{2k}^*} e^{j\omega_k n\Delta t} \ . \tag{5.2-3a}$$

But from Appendix I, we have the relation

$$\overline{\alpha_{1k} \alpha_{2k}^*} = \sum_{i=1}^{J} S_{ki} e^{-j\omega_k \tau_i} \ . \tag{5.2-3b}$$

where τ_i is the true time delay between two sensors for target i and S_{ki} is the corresponding target discrete power spectral density. Substituting Equation (5.2-3b) into Equation (5.2-3a) yields

$$\overline{R}(n\Delta t) = \sum_{i=1}^{J} \sum_{k=-B}^{B} S_{ki} \ |h_k|^2 \ e^{j\omega_k(n\Delta t-\tau_i)}$$

$$= \sum_{i=1}^{J} S_i \ \rho_i(n\Delta t - \tau_i) \tag{5.2-3c}$$

where the ith target power S_i and the normalized auto-correlation are given by

$$S_i = \sum_{k=-B}^{B} S_{ki} \tag{5.2-3d}$$

$$\rho_i(n\Delta t - \tau_i) = \sum_{k=-B}^{B} (S_{ki}/S_i) \ |h_k|^2 \ e^{j\omega_k(n\Delta t-\tau_i)} \ . \tag{5.2-3e}$$

The covariance of the random component is given by (see Appendix I)

$$\Lambda_{nm} = E[W(n\Delta t) \ W(m\Delta t) - \overline{W(n\Delta t) \ W(m\Delta t)}]$$

$$= \sum_{k=-B}^{B} |h_k|^4 \left[G_{11}(k) \ G_{22}(k) \ e^{-j\omega_k m\Delta t} + G_{12}^2(k) \ e^{j\omega_k m\Delta t}\right] e^{j\omega_k n\Delta t} \tag{5.2-4a}$$

where $G_{11}(k)$, $G_{22}(k)$, and $G_{12}(k)$ are the discrete auto-spectra and discrete cross spectrum, respectively, given by:

$$G_{11}(k) = \overline{\alpha_{1k} \ \alpha_{1k}^*} = \sum_{i=1}^{J} S_{ki} + N_{ik} \tag{5.2-4b}$$

$$G_{22}(k) = \overline{\alpha_{2k}\ \alpha_{2k}^*} = \sum_{i=1}^{J} S_{ki} + N_{2k} \tag{5.2-4c}$$

and

$$G_{12}(k) = \overline{\alpha_{1k}\ \alpha_{2k}^*} = \sum_{i=1}^{J} S_{ki}\ e^{-j\omega_k \tau_i} \ . \tag{5.2-4d}$$

Now define the 2J unknown parameter vector by

$$\underline{\theta} = (S_1, S_2, \ldots, S_J; \tau_1, \tau_2, \ldots, \tau_J)^T \tag{5.2-5a}$$

and the assumed matching function by

$$h_n(\underline{\theta}) = \sum_{i=1}^{J} S_i\ \rho_i(n\Delta t - \tau_i) \ , \tag{5.2-5b}$$

then the observation Equation (5.2-2b) can be written as

$$R(n\Delta t) = h_n(\underline{\theta}) + W(n\Delta t) \ ; \qquad n = 0, \pm 1, \pm 2, \ldots, \pm N/2 \ . \tag{5.2-5c}$$

In matrix notation, this can be written as

$$\underline{Z} = \underline{h}(\underline{\theta}) + \underline{W} \tag{5.2-6a}$$

where

$$\underline{Z} = [R(-N\Delta t/2), \ldots, R(N\Delta t/2)]^T \tag{5.2-6b}$$

$$\underline{h}(\underline{\theta}) = \left[h_{-N/2}(\underline{\theta}), \ \ldots, \ h_{-N/2}(\underline{\theta}) \right]^T \tag{5.2-6c}$$

$$\underline{W} = [W(-N\Delta t/2), \ldots, W(N\Delta t/2)]^T \tag{5.2-6d}$$

are N+1 dimensional vectors.

Note that the noise vector is zero mean with matrix covariance

$$E(\underline{W}\,\underline{W}^T) = \Lambda \tag{5.2-6e}$$

where the mn element of Λ is given by Equation (5.2-4a). Using an LMS error criteria, the best estimate of the unknown parameter vector $\underline{\theta}$ is obtained by minimizing the function

$$J(\underline{\theta}) = [\underline{Z} - \underline{h}(\underline{\theta})]^T\ [\underline{Z} - \underline{h}(\underline{\theta})] \ . \tag{5.2-7}$$

Thus the best estimate of $\underline{\theta}$ is given by

$$\hat{\underline{\theta}} = \underset{\underline{\theta}}{\text{Min Arg}}\ J(\underline{\theta}) \tag{5.2-8a}$$

or equivalently the vector null equation

$$\left.\frac{\partial J(\underline{\theta})}{\partial \underline{\theta}}\right|_{\underline{\theta} = \hat{\underline{\theta}}} = \left[\frac{\partial \underline{h}(\underline{\theta})}{\partial \underline{\theta}}\right]^T_{\underline{\theta} = \hat{\underline{\theta}}} \left[\underline{Z} - \underline{h}(\hat{\underline{\theta}})\right] = \underline{0} \ . \tag{5.2-8b}$$

Note that because we are trying to find a best match of the observed GCC output to an assumed reference function, the resulting estimator is appropriately called the Matched Estimator.

5.3 ESTIMATOR PERFORMANCE EVALUATION

In this section we present the multi-parameter covariance matrix bound for the post GCC multitarget processor. Rewriting the vector null equation (5.2-8b) as

$$\underline{y}(\underline{\theta}) = \left[\frac{\partial \underline{h}(\underline{\theta})}{\partial \underline{\theta}}\right]^T \left[\underline{Z} - \underline{h}(\underline{\theta})\right] = \underline{0} \,, \tag{5.3-1}$$

the Taylor series expansion of $\underline{y}(\underline{\theta})$ about $\underline{\theta}_0$, the true parameter vector yields (ignoring the higher order terms)

$$\underline{y}(\underline{\theta}) = \underline{y}(\theta_0) + \left.\frac{\partial \underline{y}(\theta)}{\partial \underline{\theta}}\right|_{\underline{\theta} = \underline{\theta}_0} (\underline{\theta} - \underline{\theta}_0) \,. \tag{5.3-2a}$$

Since by definition $\underline{y}(\hat{\underline{\theta}})=\underline{0}$, we must have

$$\underline{y}(\underline{\theta}_0) = - \left.\frac{\partial \underline{y}(\theta)}{\partial \underline{\theta}}\right|_{\underline{\theta} = \underline{\theta}_0} (\hat{\underline{\theta}} - \underline{\theta}_0)$$

$$= - A(\theta_0)\, \delta\hat{\underline{\theta}} \tag{5.3-2b}$$

where $\delta\hat{\underline{\theta}} \triangleq (\hat{\underline{\theta}} - \underline{\theta}_0)$ and

$$A(\theta_0) = \left.\frac{\partial \underline{y}(\theta)}{\partial \underline{\theta}}\right|_{\underline{\theta} = \underline{\theta}_0}$$

$$= \left.\frac{\partial \underline{h}(\underline{\theta})}{\partial \underline{\theta}}\right|_{\underline{\theta} = \underline{\theta}_0} [Z - \underline{h}(\theta_0)] - \left[\frac{\partial \underline{h}(\underline{\theta})}{\partial \underline{\theta}}\right]^T_{\underline{\theta} = \underline{\theta}_0} \left[\frac{\partial \underline{h}(\theta)}{\partial \underline{\theta}}\right]_{\underline{\theta} = \underline{\theta}_0}$$

$$\approx -H^T(\theta_0)\, H(\underline{\theta}_0) \tag{5.3-2c}$$

In deriving Equation (5.3-2c), it was assumed that the first term is negligible for sufficiently high SNR. Therefore, from Equation (5.3-2b), one obtains the matrix covariance equation

$$COV[\underline{y}(\underline{\theta}_0)] = A(\underline{\theta}_0)\ COV[\delta\hat{\underline{\theta}}]\ A(\underline{\theta}_0)\ . \tag{5.3-2d}$$

But from Equation (5.3-1)

$$COV[\underline{y}(\underline{\theta}_0)] = H^T(\theta_0)\ \Lambda\ H(\underline{\theta}_0)\ . \tag{5.3-2e}$$

Therefore, the estimator's matrix covariance is

$$COV[\delta\hat{\theta}] = A^{-1}(\underline{\theta}_0)\ H^T(\underline{\theta}_0)\ \Lambda\ H(\underline{\theta}_0)\ A^{-1}(\underline{\theta}_0)\ . \tag{5.3-3}$$

5.4 SIMULATION

A computer program was developed on the VAX-11/780 at NUSC to simulate the multitarget GCC output. For simplicity, a two-sensor, two-target environment was chosen. The GCC output observation window was set at ten times the reciprocal signal bandwidth. Both targets are assumed to have a broadband flat spectra. One thousand Monte Carlo iterations were used for each time delay separation. Simulation results are plotted along with the theoretical performance predictions.

5.4.1 Simulation Procedure

Figure 5-1 diagrams the simulation procedure. The multitarget GCC output observation vector (Equation (5.2-6a)) was generated by adding observation noise sequence with prescribed covariance to the noise-free GCC component. The noise vector was generated using the following procedure: (1) calculating the multitarget GCC covariance matrix from Equation (5.2-4a), (2) factoring the matrix into lower and upper triangular matrices using the Gramm-Schmidt orthogonalization procedure, and (3) multiplying the lower triangular matrix by a white noise vector to produce the desire correlated observation noise vector. Finally, the GCC observation vector was processed by the matched estimator.

5.4.2 Discussion of Results

Since a detailed investigation of the matched estimator will be shown elsewhere (References 27, 28), only a brief discussion of the simulation results will be presented here. Figure 5-2 shows the bias characteristics of a conventional GCC estimator; the target assumed a broadband, one-sided spectrum between 0 and B Hz while the interference assumed a broadband, one-side spectrum between B/2 and B Hz, the target strength is at 0 dB and the interference is at -1 dB. Note the large bias when targets are separated just below the reciprocal signal bandwidth. In this region, targets are not resolved with the conventional GCC estimator. Figures 5-3 and 5-4 show the normalized rms (normalized by the time delay standard deviation of target only) performance of a conventional GCC estimator, the post GCC matched estimator, and the optimum multitarget estimator as a function of time delay separation. While Figure 5-3 assumed identical broadband signal and interference spectra, Figure 5-4 assumed the interference occupied the upper half frequency band but maintained the same interference power. Note that the conventional GCC rms was clipped at a degradation ratio of 20 for time delay separation less than the reciprocal signal bandwidth. Both Figure 5-3 and 5-4 show the marked improvement of the matched estimator over the GCC estimator. Note the close performance between the matched estimator and the optimum processor.

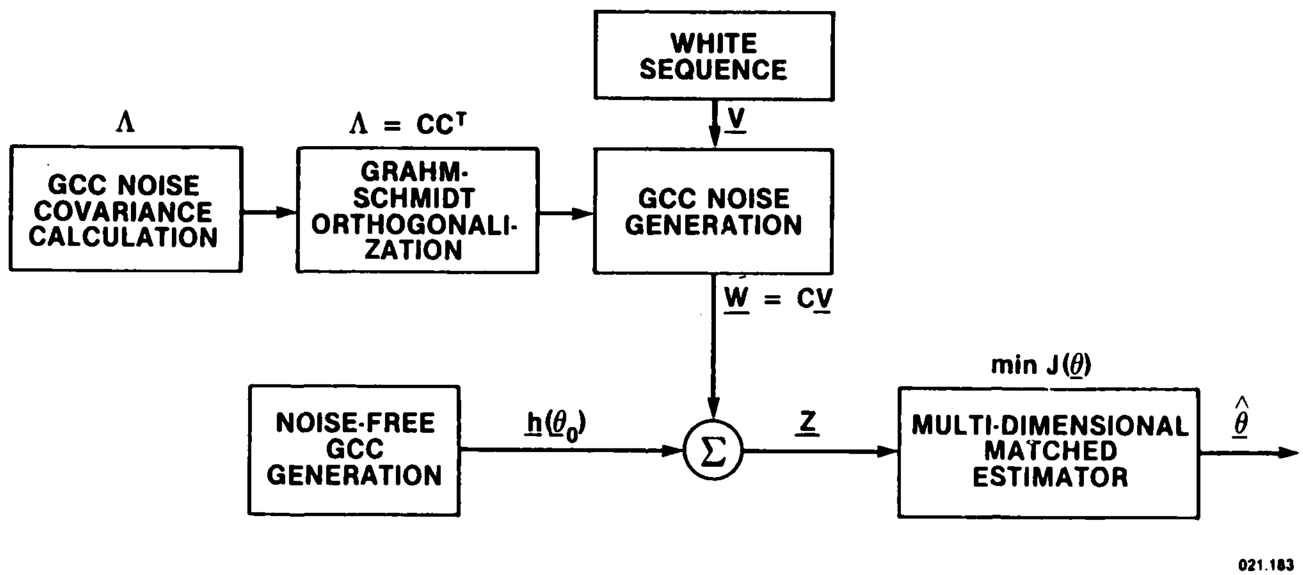


Figure 5-1. Post GCC Multitarget Processor Simulation

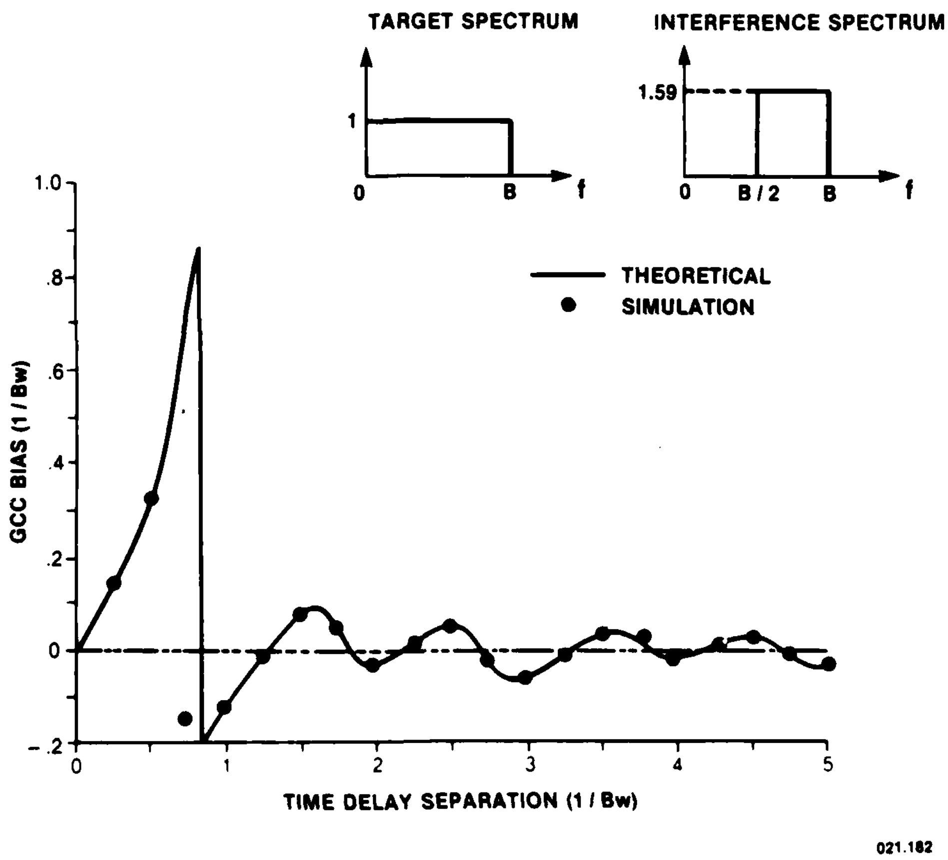


Figure 5-2. GCC Bias Versus Time-Delay Separation (SNR = 0 dB, INR = -1 dB; S:0 → B, I:B/2 → B, N:0 → B)

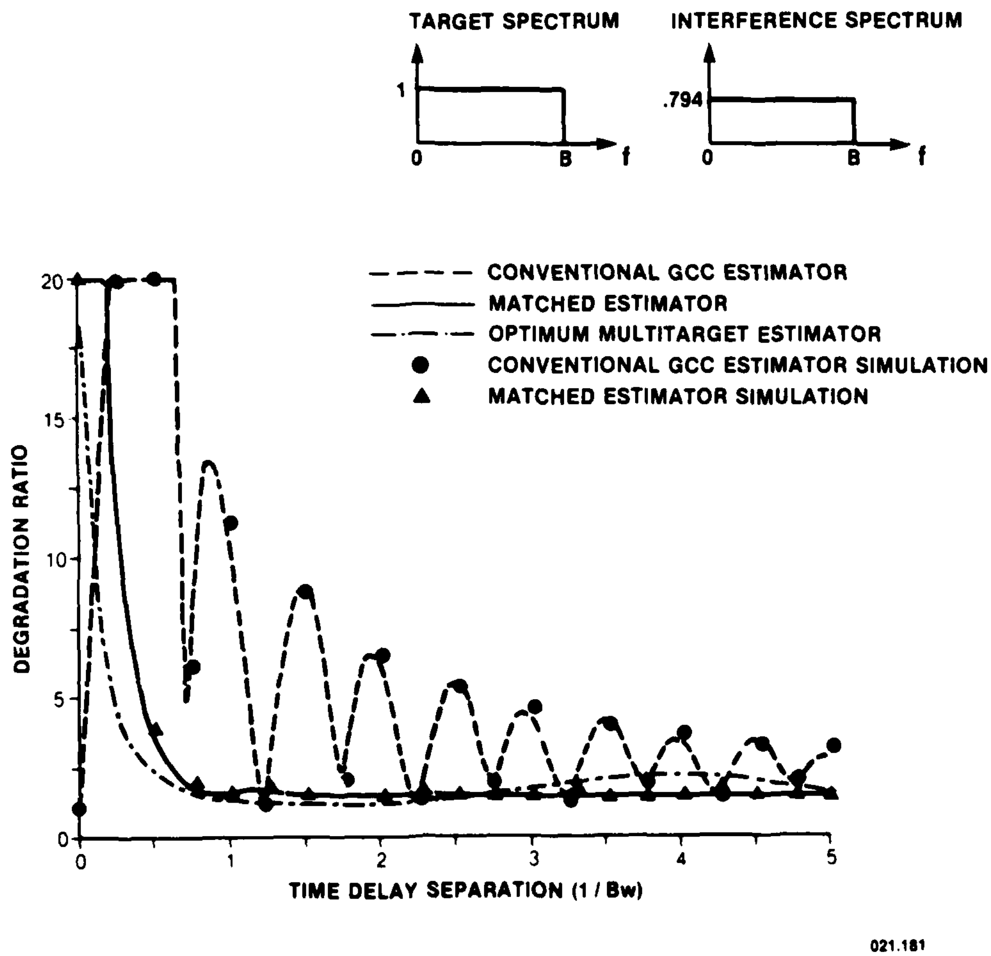


Figure 5-3. Degradation Ratio Versus Time-Delay Separation (SNR = 0 dB, INR = -1 dB; S:0 → B, I:0 → B, N:0 → B)

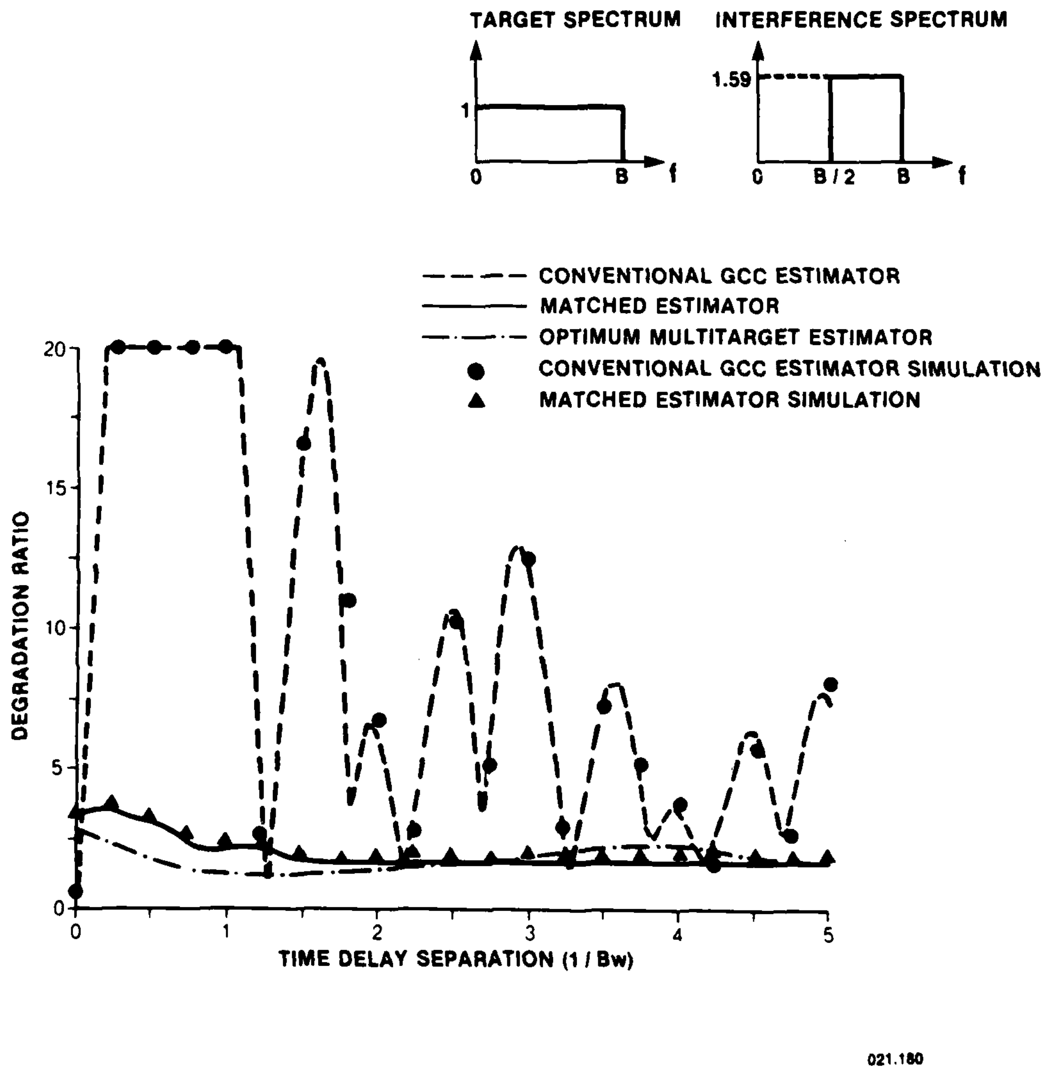


Figure 5-4. Degradation Ratio Versus Time-Delay Separation (SNR = 0 dB, INR = -1 dB; S:0 $\rightarrow$ B, I:B/2 $\rightarrow$ B, N:0 $\rightarrow$ B)

CHAPTER 6
SUMMARY AND CONCLUSIONS

We have studied in detail the methodologies of optimum signal processing for passive time delay estimation in a multisensor, multitarget environment. Our investigations were motivated by (1) the inherent inability of the existing time delay estimator to resolve estimation bias in a multitarget environment, and (2) the apparent lack of research and understanding in this area. In the literature, the studies of interference were confined to studying the effect of interference on the existing processor and the methods of interference suppression. The location or direction of interference was usually assumed known. Our study as presented here, however, treats the interference as another target of interest.

We pointed out that the traditional approach is optimum for a single target only. In the multitarget environment, the existing approach is biased due to the inherent mismatch between the single target signal processor design and the multitarget operating environment. We argue that for a high performance sonar system, it is important to have an unbiased processor since measurement bias cannot be easily removed with further post-estimator processing.

The optimum multisensor, multitarget time delay processor is derived from a Maximum Likelihood viewpoint. The actual processor is obtained by reducing the likelihood equation via straightforward but somewhat tedious manipulations to the simplest form. The general multisensor, multitarget processor is applied to a two-sensor, one-target case. The resulting processor is shown to be identical to the GCC studied by Knapp and Carter (Reference 1). We next studied the two-sensor, two-target problem. We showed the optimum two-sensor, two-target processor and presented the CRLB. We then extended our study to address the estimation of localization parameters. Much of our attention had been spent on the discussion of a basic three-sensor ranging array. We showed that for passive ranging and directional finding, an optimum processor is the focus beamformer which yields a direct estimate of range and bearing. We argued that an alternate approach is to measure the

inter-sensor time delay and then geometrically map the time delay measurements to the corresponding ranges and bearings.

The focus beamformer approach can be called the one-step approach, whereas the alternate time delay approach can be called the two-step approach. It was shown in Section 3.5.2 that both approaches yield identical performance in terms of the CRLB. However, there are major differences between the two approaches: (1) the focus beamformer requires searching over a range/bearing space of a correlation function which is asymmetric with respect to the range and bearing variables; on the other hand, any correlation over the time delay variables is always symmetrical; (2) for tracking purposes, the focus beamformer approach requires a two-dimensional error detector design whereas the time delay approach can be implemented using two one-dimensional error detectors; and (3) arguing from the Law of Large Numbers, both approaches yield Gaussian measurement noise. However, for the time delay approach, the resulting range and bearing estimates could be biased and have non-Gaussian statistics if a direct non-linear geometric mapping from time delay measurements is used. For a practical implementation, the two-step time delay approach is preferred because the symmetry of the correlation function over the time delay variables yields simple and efficient tracking logic. This is especially advantageous for a low SNR environment where the increased smoothing time required (to make the estimator efficient) can be achieved via a simple feedback design. Finally, for the three-sensor ranging array, the potential bias and non-Gaussian statistics can be minimized if mapping to intermediate variables is used instead of range and bearing; for example, mapping to cosine bearing and inverse range since they are linearly related to time delay measurements.

We also investigated the general optimum inter-sensor time delay vector estimator. Our study showed that given M sensors, the M-1 time delay can be obtained with M-1 correlators. This is in marked contrast with Hahn's approach (Reference 25) where a total M(M-1)/2 correlations are required. In addition, we presented a simple expression of the CRLB of time delay vector estimation under a single target assumption. Finally, in our study of the three-sensor ranging array, we pointed out the improved performance of the optimum formulation versus the conventional approach.

We stated that one of the strongest assumptions we made in studying the optimum time delay processor is the assumption of known target power spectrum. Therefore, we briefly addressed the problem of power spectral estimation. We studied the optimum spectral estimator for the two-sensor, one-target case and the two-sensor, two-target case. We briefly studied the problem of joint time delay and spectral estimation. A somewhat surprising result is that time delay estimates and spectral estimates are uncorrelated. This implies that the joint time delay/spectral estimation does not degrade the resulting estimator performance when assuming either is known. Furthermore, we found that while the time delay CRLB decreases as the inverse of observation time, the power spectral CRLB decreases as the inverse square of observation time.

We next addressed the problem of practical implementation of the optimum multisensor, multitarget time delay processor. We noted that one approach is to assume a weak signal in noise environment. Here the intricate structure of the optimum multisensor, multitarget time delay processor is drastically reduced to a manageable form. We also discussed a single target assumption approach as a suboptimal processor in a multitarget environment. We provided a numerical illustration of the single target processor behavior in the presence of interference. Performance of the single target processor was compared to the optimum multitarget processor. We noted that in the presence of interference, the single target processor, in general, was biased and had a larger variance except when the target interference time delay separation is small; i.e., less than a correlation pulse width. Within this region, however, the optimum multitarget processor has a variance which grows without bound as the separation decreases for the case of identical signal and interference spectrum. This reflected the inappropriateness of using a multitarget formulation in a single (merged) target environment.

In a multisensor, multitarget environment, an optimum processor remains optimum so long as the actual number of sensors and targets matches the number assumed in the optimum processor design. A mismatch in either the number of targets (addressed in this study) or the number of sensors (caused perhaps by element failure) will automatically degrade the processor performance. Therefore, a key element in using the optimum multitarget processor is the correct detection of the number of targets.

Last but not least, the optimum processors which we have derived and studied are based on a SPLOT assumption. For a moving target, this assumption is difficult to satisfy. Therefore, the optimum multitarget processor presented in this study must be further refined to account for the moving target environment.

APPENDIX A

DEFINITION OF COMPLEX GAUSSIAN PROBABILITY DENSITY FUNCTION (pdf)

This appendix briefly describes the meaning of a complex Gaussian probability density function (pdf). A thorough treatment of this subject can be found in Goodman (Reference 26).

Let X(t) be a zero mean wide-sense stationary white Gaussian random process with correlation function $E[X(t)\ X(\tau)] = \sigma^2\delta(t - \tau)$, then its Fourier coefficients from a T-second observation are given by

$$X_k = \frac{1}{T} \int_0^T X(t)e^{-j\omega_k \tau}\, dt$$

$$= I_k - j\ Q_k\ ; \qquad k = 1, 2, \ldots, B \tag{A-1}$$

where I_k and Q_k are known as the in-phase and quadrature phase components. It can be easily verified that I_k and Q_k are Gaussian distributed with the following statistics:

$$E(I_k) = E(Q_k) = E(I_k\ Q_k) = 0 \tag{A-2a}$$

$$E(I_k^2) = E(Q_k^2) = \frac{\sigma^2}{2T}\ . \tag{A-2b}$$

Let $\underline{Z}_k = (I_k\ Q_k)^T$, then the bivariate real Gaussian pdf of $\underline{Z}_k$ is given by

$$P(\underline{Z}_k) = (2\pi)^{-1}\ |R_k|^{-\frac{1}{2}} \exp\{-\ \frac{1}{2}\ \underline{Z}_k^T\ R_k^{-1}\ \underline{Z}_k\} \tag{A-3a}$$

where

$$R_k \triangleq E\ (\underline{Z}_k\ \underline{Z}_k{}^T) = \frac{1}{2T}\begin{bmatrix}\sigma^2 & 0\\ 0 & \sigma^2\end{bmatrix} \tag{A-3b}$$

Now a complex Gaussian uni-variate pdf of X_k is defined by

$$P(X_k) = \pi^{-1} |c_k|^{-1} \exp\{-X_k^* c_k^{-1} X_k\} \tag{A -4a}$$

where

$$c_k = E(X_k X_k^*) = \sigma^2/T . \tag{A-4b}$$

It is straightforward to verify that $|R_k|^{-\frac{1}{2}} = 2|c_k|^{-1}$ and $X_k^* c_k^{-1} X_k = \underline{Z}_k^T R_k^{-1} \underline{Z}_k/2$, hence one can write

$$P(X_k) = P(\underline{Z}_k) = (\pi\sigma^2)^{-1} \exp\left\{- \frac{I_k^2 + Q_k^2}{\sigma^2}\right\}. \tag{A-5}$$

Therefore, in general let $\underline{X}$ be a complex Gaussian B-dimensional vector such that

$$\underline{X} = \underline{I} - j \underline{Q} \tag{A-6a}$$

with

$$E(\underline{I}\ \underline{I}^T) = E(\underline{Q}\ \underline{Q}_T) = V/2 \tag{A-6b}$$

$$E(\underline{I}\ \underline{Q}^T) = -E(\underline{Q}\ \underline{I}^T) = -W/2 . \tag{A-6c}$$

Now define a 2B-dimensional real vector as

$$\underline{Z} = (\underline{I}^T\ \underline{Q}^T)^T . \tag{A-7}$$

Then it can be shown that $P(\underline{X})$ is a complex Gaussian pdf defined by

$$P(\underline{X}) = P(\underline{Z}) = \pi^{-B} |c|^{-1} \exp\{-\underline{X}^* c^{-1} \underline{X}\} \tag{A-8}$$

where

$$c = E(\underline{X}\ \underline{X}^*) = V + j\,W\ . \tag{A-9}$$

APPENDIX B

CALCULATION OF CRAMER-RAO LOWER BOUND FOR TWO-SENSOR, TWO-TARGET CASE

This appendix calculates the Cramer-Rao Lower Bound (CRLB) for the two-sensor, two-target case. Using Equations (3.4-1) and (3.4-5), the CRLB of the time delay estimates are:

$$\mathrm{VAR}(\hat{\tau}_i) \geq [J^{-1}]_{ii} \; ; \qquad i = 1, 2 \,. \tag{B-1}$$

Now the symmetric Fisher's Information matrix J can be written as

$$J^{-1} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}^{-1} = \frac{1}{(J_{11}J_{22} - J_{12}^2)} \begin{bmatrix} J_{22} & -J_{12} \\ -J_{12} & J_{11} \end{bmatrix} .$$

Therefore

$$\mathrm{VAR}(\hat{\tau}_i) \geq \frac{1}{(1 - M_{12}^2)} \frac{1}{J_{ii}} \; ; \; i = 1, 2 \tag{B-2}$$

where

$$M_{12} = \frac{J_{12}}{(J_{11} \; J_{22})^{\frac{1}{2}}}$$

is defined as the coefficient of mutual dependence. The quantities J_{ij} are defined by Equation (3.4-5) as:

$$J_{ij} = \sum_{k=1}^{B} \mathrm{tr}\left(- \frac{\partial R_k^{-1}}{\partial \tau_i} \frac{\partial R_k}{\partial \tau_j}\right) ; \qquad \begin{array}{l} i = 1, 2 \\ j = 1, 2 \end{array} \tag{B-3}$$

In what follows, we present a detailed calculation of the quantity J_{ij}. Using the relations given in Equations (3.2-6) and (3.5.1-4a); i.e.,

$$R_k = S_{k1} P_{k1} + S_{k2} P_{k2} + N_k I \tag{B-4}$$

and

$$- \frac{\partial R_k^{-1}}{\partial \tau_i} = |h_{ki}|^2 \tilde{Q}_{ki}^{-1} \left(\frac{\partial P_{ki}}{\partial \tau_i} - \tilde{a}_{ki} \frac{\partial G_{ki}}{\partial \tau_i} P_{ki}\right) \tilde{Q}_{ki}^{-1} ; \quad i = 1, 2 \tag{B-5}$$

we obtain

$$\mathrm{tr}\left(- \frac{\partial R_k^{-1}}{\partial \tau_i} \frac{\partial R_k}{\partial \tau_j}\right) = S_{kj} |\tilde{h}_{ki}|^2 \left[\mathrm{tr}\left(\tilde{Q}_{ki}^{-1} \frac{\partial P_{ki}}{\partial \tau_i} \tilde{Q}_{ki}^{-1} \frac{\partial P_{kj}}{\partial \tau_j}\right)\right.$$

$$\left. - \tilde{a}_{ki} \frac{\partial G_{ki}}{\partial \tau_i} \mathrm{tr}\left(\tilde{Q}_{ki}^{-1} P_{ki} \tilde{Q}_{ki}^{-1} \frac{\partial P_{kj}}{\partial \tau_j}\right)\right] . \tag{B-6}$$

The trace of the quantities inside the parentheses can be evaluated as follows. From Equation (3.2-7c), we find

$$\tilde{Q}_{k1}^{-1} = \tilde{N}_{k1}(S_{k2} P_{k2} + N_k I)^{-1}$$

$$= \frac{\tilde{N}_{k1}}{N_k} \left[I - \frac{S_{k2}/N_k}{1 + 2 S_{k2}/N_k} P_{k2}\right]$$

$$= \frac{\tilde{N}_{k1}}{N_k} \begin{bmatrix} 1 - a_{k2} & -a_{k2}\, e^{-j\omega_k \tau_2} \\ -a_{k2}\, e^{j\omega_k \tau_2} & 1 - a_{k2} \end{bmatrix} \tag{B-7}$$

where

$$a_{k2} = \frac{S_{k2}/N_k}{1 + 2\, S_{k2}/N_k} \tag{B-8}$$

and similarly

$$\tilde{Q}_{k2}^{-1} = \frac{\tilde{N}_{k2}}{N_k} \begin{bmatrix} 1 - a_{k1} & -a_{k1}\, e^{-j\omega_k \tau_1} \\ -a_{k1}\, e^{j\omega_k \tau_1} & 1 - a_{k1} \end{bmatrix} \tag{B-9}$$

where

$$a_{k1} = \frac{S_{k1}/N_k}{1 + 2\, S_{k1}/N_k} \quad . \tag{B-10}$$

Note that $\tilde{Q}_{ki}^{-1}$ is a Hermitian matrix with identical diagonal elements. Therefore, letting

$$\tilde{Q}_{ki}^{-1} = \begin{bmatrix} q_{11}^{i} & q_{12}^{i} \\ q_{12}^{i*} & q_{11}^{i} \end{bmatrix}$$

we have

$$\tilde{Q}_{ki}\,\frac{\partial P_{kj}}{\partial \tau_j} = j\omega_k \begin{bmatrix} q_{11}^{i} & q_{12}^{i} \\ q_{12}^{i*} & q_{11}^{i} \end{bmatrix} \begin{bmatrix} 0 & -e^{-j\omega_k \tau_j} \\ e^{j\omega_k \tau_j} & 0 \end{bmatrix}$$

$$= j\omega_k \begin{bmatrix} q_{12}^{i}\, e^{j\omega_k \tau_j} & -q_{11}^{i}\, e^{-j\omega_k \tau_j} \\ q_{11}^{i}\, e^{j\omega_k \tau_j} & -q_{12}^{i*}\, e^{-j\omega_k \tau_j} \end{bmatrix} \qquad \text{(B-11)}$$

and after some algebraic manipulation, we obtain from Equation (B-11)

$$\tilde{Q}_{ki}^{-1}\,\frac{\partial P_{ki}}{\partial \tau_i}\,\tilde{Q}_{ki}^{-1}\,\frac{\partial P_{kj}}{\partial \tau_j} = -\omega_k^2 \begin{bmatrix} A_{11}^{ij} & A_{12}^{ij} \\ A_{12}^{ij*} & A_{11}^{ij*} \end{bmatrix} \qquad \text{(B-12)}$$

where

$$A_{11}^{ij} = (q_{12}^{i})^2\, e^{j\omega_k(\tau_i+\tau_j)} - (q_{11}^{i})^2\, e^{-j\omega_k(\tau_i-\tau_j)}$$

$$A_{12}^{ij} = (q_{11}^{i}) \left[q_{12}^{i}\, e^{j\omega_k(\tau_i+\tau_j)} - q_{12}^{*}\, e^{-j\omega_k(\tau_i-\tau_j)} \right] .$$

Thus

$$\mathrm{tr} \left(\tilde{Q}_{ki}^{-1}\,\frac{\partial P_{ki}}{\partial \tau_i}\,\tilde{Q}_{ki}^{-1}\,\frac{\partial P_{kj}}{\partial \tau_j} \right) = -\omega_k^2\, (A_{11}^{ij} + A_{11}^{ij*})$$

$$= -2\omega_k^2\, \mathrm{Re}\, \{A_{11}^{ij}\} \qquad \text{(B-13)}$$

where Re { } denotes the real part.

Now using Equations (B-7) and (B-8), we obtain

$$\mathrm{tr}\left[\left(\tilde{Q}_{k1}^{-1}\,\frac{\partial P_{k1}}{\partial \tau_1}\right)^2\right] = 2\omega_k^2\left(\frac{\tilde{N}_{k1}}{N_k}\right)^2 (1 - a_{k2})^2\left[1 - \frac{a_{k2}^2}{(1 - a_{k2})^2}\cos 2\omega_k\Delta_{12}\right] \tag{B-14a}$$

$$\mathrm{tr}\left[\left(\tilde{Q}_{k2}^{-1}\,\frac{\partial P_{k2}}{\partial \tau_2}\right)^2\right] = 2\omega_k^2\left(\frac{\tilde{N}_{k2}}{N_k}\right)^2 (1 - a_{k1})^2\left[1 - \frac{a_{k2}^2}{(1 - a_{k1})^2}\cos 2\omega_k\Delta_{12}\right] \tag{B-14b}$$

$$\mathrm{tr}\left(\tilde{Q}_{k1}^{-1}\,\frac{\partial P_{k1}}{\partial \tau_1}\,\tilde{Q}_{k1}^{-1}\,\frac{\partial P_{k2}}{\partial \tau_2}\right) = 2\omega_k^2\left(\frac{\tilde{N}_{k1}}{N_k}\right)^2 (1 - 2a_{k2})\cos \omega_k\Delta_{12} \tag{B-14c}$$

where

$$\Delta_{12} \triangleq \tau_1 - \tau_2$$

is the time delay separation.

On the other hand, we have

$$\tilde{Q}_{k1}^{-1}\,P_{ki} = \begin{bmatrix} q_{11}^{i} & q_{12}^{i} \\ q_{12}^{i*} & q_{11}^{i} \end{bmatrix} \begin{bmatrix} 1 & e^{-j\omega_k\tau_i} \\ e^{j\omega_k\tau_i} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} q_{11}^{i} + q_{12}^{i} e^{j\omega_j \tau_i} & q_{11}^{i} e^{-j\omega_k \tau_i} + q_{12}^{i} \\ q_{12}^{i*} + q_{11}^{i} e^{j\omega_k \tau_i} & q_{12}^{i*} e^{-j\omega_k \tau_i} + q_{11}^{i} \end{bmatrix}$$

$$= \begin{bmatrix} B_{11}^{i} & B_{12}^{i} \\ B_{12}^{i} & B_{11}^{i*} \end{bmatrix} . \tag{B-15}$$

Thus combining Equations (B-11) and (B-15) we obtain

$$\mathrm{tr}\left(\tilde{Q}_{ki}^{-1} P_{ki} \tilde{Q}_{ki}^{-1} \frac{\partial P_{kj}}{\partial \tau_j}\right) = j\omega_k \ \mathrm{tr}\left\{\begin{bmatrix} B_{11}^{i} & B_{12}^{i} \\ B_{12}^{i*} & B_{11}^{i*} \end{bmatrix} \begin{bmatrix} q_{12}^{i} e^{j\omega_k \tau_j} - q_{11}^{i} e^{-j\omega_k \tau_j} \\ q_{11}^{i} e^{j\omega_k \tau_j} - q_{12}^{i*} e^{-j\omega_k \tau_j} \end{bmatrix}\right\}$$

$$= j\omega_k \Big[(B_{11}^{i} q_{12}^{i} + B_{12}^{i} q_{11}^{i}) e^{j\omega_k \tau_j}$$

$$- (B_{12}^{i*} q_{11}^{i} + B_{11}^{i*} q_{12}^{i}) e^{-j\omega_k \tau_j} \Big]$$

$$= j\omega_k [C_{ij} - C_{ij}^{*}]$$

$$= -2\omega_k \ \mathrm{Im}\{C_{ij}\} \tag{B-16}$$

where

$$C_{ij} \triangleq (B_{11}^{i} q_{12}^{i} + B_{12}^{i} q_{11}^{i}) e^{j\omega_k \tau_j}$$

$$= (q_{11}^{i})^2 e^{-j\omega_k \Delta_{ij}} + (q_{12}^{i})^2 e^{j\omega_k \sigma_{ij}} + 2 q_{11}^{i} q_{12}^{i} e^{j\omega_k \tau_j} , \tag{B-17}$$

Im { } denotes the imaginary part,

and $\Delta_{ij} \triangleq \tau_i - \tau_j \; ; \qquad \sigma_{ij} \triangleq \tau_i + \tau_j \; .$

Now from Equations (B-7) and (B-8), the following are obtained:

$$C_{11} = \left(\frac{\tilde{N}_{k1}}{N_k}\right)^2 (1 - a_{k2})^2 \left(1 - \frac{a_{k2}}{1 - a_{k2}} e^{j\omega_k \Delta_{12}}\right)^2 \tag{B-18a}$$

$$C_{22} = \left(\frac{\tilde{N}_{k2}}{N_k}\right)^2 (1 - a_{k1})^2 \left(1 - \frac{a_{k1}}{1 - a_{k1}} e^{j\omega_k \Delta_{21}}\right)^2 \tag{B-18b}$$

and

$$C_{12} = \left(\frac{\tilde{N}_{k1}}{N_k}\right)^2 (1 - a_{k2})^2 \left(1 - \frac{a_{k2}}{1 - a_{k2}} e^{j\omega_k \Delta_{12}}\right)^2 e^{-j\omega_k \Delta_{12}} \quad . \tag{B-18c}$$

Now substituting Equations (B-18a-c) into (B-16) yields

$$\mathrm{tr}\left(\tilde{Q}_{k1}^{-1} P_{k1} \tilde{Q}_{k1}^{-1} \frac{\partial P_{k1}}{\partial \tau_1}\right) = 2\omega_k \left(\frac{\tilde{N}_{k1}}{N_k}\right)^2 a_{k2} (1 - a_{k2})$$

$$\left(2 \sin \omega_k \Delta_{12} - \frac{a_{k2}}{1 - a_{k2}} \sin 2\omega_k \Delta_{12}\right) \tag{B-19a}$$

$$\mathrm{tr}\left(\tilde{Q}_{k2}^{-1}\, P_{k2}\, \tilde{Q}_{k2}^{-1}\, \frac{\partial P_{k2}}{\partial \tau_2}\right) = 2\omega_k \left(\frac{\tilde{N}_{k2}}{N_k}\right)^2 a_{k1}\, (1 - a_{k1})$$

$$\left(2 \sin \omega_k \Delta_{12} - \frac{a_{k2}}{1 - a_{k2}} \sin 2\omega_k \Delta_{12}\right) \qquad \text{(B-19b)}$$

and

$$\mathrm{tr}\left(\tilde{Q}_{k1}^{-1}\, P_{k1}\, \tilde{Q}_{k1}^{-1}\, \frac{\partial P_{k2}}{\partial \tau_2}\right) = 2\omega_k \left(\frac{\tilde{N}_{k1}}{N_k}\right)^2 (1 - 2a_{k2})\; 2 \sin \omega_k \Delta_{12} \; . \qquad \text{(B-19c)}$$

In addition, from Equations (3.5.1-5b) and (B-11), we have

$$\frac{\partial G_{ki}}{\partial \tau_i} = \mathrm{tr}\left(\tilde{Q}_{ki}^{-1}\, \frac{\partial P_{ki}}{\partial \tau_i}\right)$$

$$= j\omega_k \left(q_{12}^{i}\, e^{j\omega_k \tau_i} - q_{12}^{i*}\, e^{-j\omega_k \tau_i}\right) \qquad \text{(B-20)}$$

and for i = 1

$$\frac{\partial G_{k1}}{\partial \tau_1} = 2\omega_k\; a_{k2} \left(\frac{\tilde{N}_{k1}}{N_k}\right) \sin \omega_k \Delta_{12} \; . \qquad \text{(B-21)}$$

Finally, substituting Equations (B-14a-c), (B-18a-c) and (B-19a-c) into (B-6) we find:

$$\mathrm{tr}\left(-\frac{\partial R_k^{-1}}{\partial \tau_1}\frac{\partial R_k}{\partial \tau_1}\right) = S_{k1}\ |h_{k1}|^2 \left\{ \mathrm{tr}\left[\left(\tilde{Q}_{k1}^{-1}\frac{\partial P_{k1}}{\partial \tau_1}\right)^2\right]\right.$$

$$\left. - \tilde{a}_{k1}\frac{\partial G_{k1}}{\partial \tau_1}\,\mathrm{tr}\left(\tilde{Q}_{k1}^{-1}\ P_{k1}\ \tilde{Q}_{k1}^{-1}\frac{\partial P_{k1}}{\partial \tau_1}\right)\right\}$$

$$= 2\omega_k^2\ S_{k1}\ |h_{k1}|^2\left(\frac{\tilde{N}_{k1}}{N_k}\right)^2 (1 - a_{k2})^2\ \gamma_1 \tag{B-22a}$$

where

$$\gamma_1 = \left\{1 - E_1 \cos 2\omega_k\Delta_{12} - E_2 \sin^2 \omega_k\Delta_{12}\right.$$

$$\left. + E_3 (\sin \omega_k\Delta_{12}) (\sin 2\omega_k\Delta_{12})\right\} \tag{B-22b}$$

with

$$E_1 = \frac{a_{k2}^2}{(1 - a_{k2})^2} \tag{B-22c}$$

$$E_2 = 4\ \tilde{a}_{k1}\ (1 - a_{k2})\left(\frac{\tilde{N}_{k1}}{N_k}\right) E_1 \tag{B-22d}$$

$$E_3 = \frac{2\ \tilde{a}_{k1}\ a_{k2}^3}{(1 - a_{k2})^2}\left(\frac{\tilde{N}_{k1}}{N_k}\right). \tag{B-22e}$$

Similarly, we have

$$tr\left(- \frac{\partial R_k^{-1}}{\partial \tau_2} \frac{\partial R_k}{\partial \tau_2}\right) = 2\omega_k^2 \; S_{k2} \; |h_{k2}|^2 \left(\frac{\tilde{N}_{k2}}{N_k}\right) (1 - a_{k1})^2 \; \gamma_2 \tag{B-23}$$

where γ_2 is obtained by exchanging indices between 1 and 2 from γ_1. Finally, the cross term is

$$tr\left(- \frac{\partial R_k^{-1}}{\partial \tau_1} \frac{\partial R_k}{\partial \tau_2}\right) = S_{k2} \; |h_{k1}|^2 \left\{ tr\left(\tilde{Q}_{k1}^{-1} \frac{\partial P_{k1}}{\partial \tau_1} \tilde{Q}_{k1}^{-1} \frac{\partial P_{k2}}{\partial \tau_2}\right)\right.$$

$$\left. - \tilde{a}_{k1} \frac{\partial G_{k1}}{\partial \tau_1} \, tr\left(\tilde{Q}_{k1}^{-1} \; P_{k1} \; \tilde{Q}_{k1}^{-1} \frac{\partial P_{k2}}{\partial \tau_2}\right) \right\}$$

$$= 2\omega_k^2 \; S_{k2} \; |h_{k1}|^2 \; (1 - 2 \; a_{k2}) \left(\frac{\tilde{N}_{k1}}{N}\right)^2 \; \gamma_{12} \tag{B-24a}$$

where

$$\gamma_{12} = \cos \omega_k \Delta_{12} - 2 \; \tilde{a}_{k1} \; a_{k2} \left(\frac{\tilde{N}_{k1}}{N_k}\right) \sin^2 \omega_k \Delta_{12} \; . \tag{B-24b}$$

Thus, the elements of the Fisher's Information matrix are:

$$J_{11} = 2 \sum_{k=1}^{B} \omega_k^2 \; S_{k1} \; |h_{k1}|^2 \left(\frac{\tilde{N}_{k1}}{N_k}\right)^2 (1 - a_{k2})^2 \; \gamma_1 \tag{B-25a}$$

$$J_{12} = 2 \sum_{k=1}^{B} \omega_k^2 \; S_{k2} \; |h_{k1}|^2 \left(\frac{\tilde{N}_{k1}}{N_k}\right)^2 (1 - 2a_{k2}) \; \gamma_{12} \tag{B-25b}$$

$$J_{22} = 2 \sum_{k=1}^{B} \omega_k^2 \; S_{k2} \; |h_{k2}|^2 \left(\frac{\tilde{N}_{k2}}{N_k} \right)^2 \; (1 - a_{k1})^2 \; \gamma_2 \; . \tag{B-25c}$$

Using the definition of $|h_{ki}|^2$, N_{ki} and a_{ki} in Equations (3.5.1-4b), (3.5.1-2c), (B-8), and (B-10), Equations (B-25a-c) can be expressed in integral form for a sufficiently long observation time T as follows:

$$J_{11} = \frac{T}{\pi} \int_0^\infty \omega^2 \left(\frac{S_1^2/N^2}{1 + G_1 S_1/(S_2 + N)} \right) \left(\frac{1 + S_2/N}{1 + 2\; S_2/N} \right)^2 \; \gamma_1 \; d\omega \tag{B-26a}$$

$$J_{12} = \frac{T}{\pi} \int_0^\infty \omega^2 \left(\frac{S_1 S_2/N^2}{1 + G_1 S_1/(S_2 + N)} \right) \left(\frac{1}{1 + 2\; S_2/N} \right) \gamma_{12} \; d\omega \tag{B-26b}$$

and

$$J_{22} = \frac{T}{\pi} \int_0^\infty \omega^2 \left(\frac{S_2^2/N^2}{1 + G_2 \; S_2/(S_1 + N)} \right) \left(\frac{1 + S_1/N}{1 + 2\; S_1/N} \right)^2 \; \gamma_2 \; d\omega \; . \tag{B-26c}$$

where

$$G_1 = (1 + S_2/N) \left[2 - \left(\frac{S_2/N}{1 + 2\; S_2/N} \right) \left| 1 + e^{-j\omega\Delta_{12}} \right|^2 \right] \tag{B-26d}$$

$$G_2 = (1 + S_1/N) \left[2 - \left(\frac{S_1/N}{1 + 2\; S_1/N} \right) \left| 1 + e^{j\omega\Delta_{12}} \right|^2 \right] \tag{B-26e}$$

A somewhat simpler expression for γ_1, γ_{12}, and γ_2 can be obtained as follows. From Equation (B-22a-e) we obtain

$$E_1 = \frac{a_{k2}^2}{(1 - a_{k2})^2} = \frac{\left(\frac{S_2/N}{1 + 2\ S_2/N}\right)^2}{\left(\frac{1 + S_2/N}{1 + 2\ S_2/N}\right)^2} = \left(\frac{S_2/N}{1 + S_2/N}\right)^2 \tag{B-27a}$$

$$E_2 = 4\ \tilde{a}_{k1}\ (1 - a_{k2})\left(\frac{\tilde{N}_{k1}}{N_k}\right) E_1 = W_1\ E_1 \tag{B-27b}$$

where

$$W_1 = 4\left[\frac{S_1/N}{1 + G_1\ S_1/(S_2 + N)}\right]\left[\frac{1 + S_2/N}{1 + 2\ S_2/N}\right] \tag{B-27c}$$

and

$$E_3 = \frac{2\ \tilde{a}_{k1}\ a_{k2}^3}{(1 - a_{k2})^2}\left(\frac{\tilde{N}_{k1}}{N_k}\right) = 2\ \tilde{a}_{k1}\ a_{k2}\left(\frac{\tilde{N}_{k1}}{N_k}\right) E_1 = W_2\ E_1 \tag{B-27d}$$

where

$$W_2 = 2\left[\frac{S_1/N}{1 + G_1\ S_1/(S_2 + N)}\right]\left[\frac{S_2/N}{1 + 2\ S_2/N}\right] . \tag{B-27e}$$

Hence Equation (B-22b) can be written as

$$\gamma_1 = 1 - E_1 \left(\cos 2\omega_k\Delta_{12} + W_1 \sin^2 \omega_k\Delta_{12} - W_2 \sin \omega_k\Delta_{12} \sin 2\omega_k\Delta_{12} \right)$$

$$= 1 - E_1 \sum_{n=0}^{3} A_n \cos^n \omega_k\Delta_{12} \qquad \text{(B-28a)}$$

where

$$A_0 = 4 \left[\frac{S_1/N}{1 + G_1 \; S_1/(S_2 + N)} \right] \left[\frac{1 + S_2/N}{1 + 2 \; S_2/N} \right] - 1 \qquad \text{(B-28b)}$$

$$A_1 = -4 \left[\frac{S_1/N}{1 + G_1 \; S_1/(S_2 + N)} \right] \left[\frac{S_2/N}{1 + 2 \; S_2/N} \right]$$

$$A_2 = 1 - A_0$$

$$A_3 = -A_1 \; .$$

Similarly, one obtains

$$\gamma_2 = 1 - E_2 \sum_{n=0}^{3} B_n \cos^n \omega_k\Delta_{12} \qquad \text{(B-28c)}$$

where

$$B_0 = 4\left[\frac{S_2/N}{1 + G_2\ S_2/(S_1 + N)}\right]\left[\frac{1 + S_1/N}{1 + 2\ S_1/N}\right] - 1 \tag{B-28d}$$

$$B_1 = -4\left[\frac{S_2/N}{1 + G_2\ S_2/(S_1 + N)}\right]\left[\frac{S_1/N}{1 + 2\ S_1/N}\right]$$

$$B_2 = 1 - B_0$$

$$B_3 = -B_1$$

Finally, from Equation (B-24b)

$$\gamma_{12} = \cos \omega_k \Delta_{12} - 2\left[\frac{S_1/N}{1 + G_1\ S_1/(S_2 + N)}\right]\left[\frac{S_2/N}{1 + 2\ S_2/N}\right] \sin^2 \omega_k \Delta_{12}\ . \tag{B-28e}$$

APPENDIX C

CRAMER-RAO LOWER BOUND OF TIME DELAY ESTIMATION FROM MULTISENSOR ARRAY

In general, the Cramer-Rao Lower Bound (CRLB) is given by

$$\mathrm{VAR}\,(\hat{\theta}_i) \geq [J^{-1}]_{ii} \tag{C-1}$$

where J is the Fisher Information matrix whose ij element is defined by (Equation (3.4-5)).

$$J_{ij} = \sum_{k=1}^{B} \mathrm{tr}\left(-\frac{\partial R_k^{-1}}{\partial \theta_i}\frac{\partial R_k}{\partial \theta_j}\right) . \tag{C-2}$$

Under the assumptions of (1) single target, (2) spatially incoherent noise processes and (3) identical sensor array noise power spectrum, the CRLB can be evaluated easily.

From Equations (3.5.1-2) and (3.5.1-4) we obtain

$$R_k = S_k P_k + N_k Q_k , \tag{C-3}$$

where $N_k Q_k = N_k I$ using assumptions (2) and (3), and the relation

$$- \frac{\partial R_k^{-1}}{\partial \theta_i} = |h_k|^2 \frac{\partial P_k}{\partial \theta_i} \quad , \tag{C-4}$$

where

$$|h_k|^2 = \frac{S_k/N_k^2}{1 + M\, S_k/N_k} \tag{C-5}$$

and M is the number of available sensors.

From Equations (3.5.1-10b) one obtains

$$\frac{\partial P_k}{\partial \theta_i} = j\omega_k \, V_k \frac{\partial \Phi}{\partial \theta_i} V_k^* \tag{C-6}$$

where V_k, the steering matrix, is given by

$$V_k = \mathrm{diag} \left\{ e^{j\omega_k D_1} \; e^{j\omega_k D_2} \; \ldots \; e^{j\omega_k D_M} \right\} \tag{C-7}$$

and the mn element of the matrix $\frac{\partial \Phi}{\partial \theta_i}$ is

$$\left(\frac{\partial \Phi}{\partial \theta_i} \right)_{mn} = \frac{\partial}{\partial \theta_i} (D_m - D_n) \; . \tag{C-8}$$

Now since

$$\frac{\partial R_k}{\partial \theta_j} = S_k \frac{\partial P_k}{\partial \theta_j} \tag{C-9}$$

from Equation (C-3) and using Equation (C-4) in Equation (C-2), one obtains

the expression

$$\mathrm{tr}\left(-\frac{\partial R_k^{-1}}{\partial \theta_i}\frac{\partial R_k}{\partial \theta_j}\right) = S_k\ |h_k|^2\ \mathrm{tr}\left(\frac{\partial P_k}{\partial \theta_i}\frac{\partial P_k}{\partial \theta_j}\right)$$

$$= -S_k\ |h_k|^2\ \omega_k^2\ \mathrm{tr}\left(V_k\frac{\partial \Phi}{\partial \theta_i}V_k^* V_k\frac{\partial \Phi}{\partial \theta_j}V_k^*\right). \qquad \text{(C-10)}$$

Now using the relation $V_k^* V_k = I$ and $\mathrm{tr}(AB) = \mathrm{tr}(BA)$, Equation (C-10) becomes

$$\mathrm{tr}\left(-\frac{\partial R_k^{-1}}{\partial \theta_i}\frac{\partial R_k}{\partial \theta_j}\right) = -S_k\ |h_k|^2\ \omega_k^2\ \mathrm{tr}\left(\frac{\partial \Phi}{\partial \theta_i}\frac{\partial \Phi}{\partial \theta_j}\right). \qquad \text{(C-11)}$$

Therefore

$$J_{ij} = -\sum_{k=1}^{B} S_k\ |h_k|^2\ \omega_k^2\ \mathrm{tr}\left(\frac{\partial \Phi}{\partial \theta_i}\frac{\partial \Phi}{\partial \theta_j}\right)$$

$$= -\mathrm{tr}\left(\frac{\partial \Phi}{\partial \theta_i}\frac{\partial \Phi}{\partial \theta_j}\right)\sum_{k=1}^{B}\omega_k^2\ \frac{S_k^2/N_k^2}{1 + M\ S_k/N_k}\ . \qquad \text{(C-12)}$$

Knowing J_{ij}'s for all i and j, Equation (C-1) can be used to calculate the CRLB. Using Equations (C-1) and (C-12), we shall establish the following relation.

Given M sensors under a single target, spatially incoherent and spectrally identically distributed noise environment, the CRLB for the incremental time delay are identical and equal to

$$\mathrm{VAR}(\hat{\tau}_i) \geq \left[M \sum_{k=1}^{B} \omega_k^2 \left(\frac{S_k^2/N_k^2}{1 + M\ S_k/N_k} \right) \right]^{-1} \quad ; \quad i = 1, 2, \ldots, M-1 \ . \qquad \text{(C-13a)}$$

Furthermore, denote the time delay between any two sensors by D, then

$$\mathrm{VAR}(\hat{D}_{ML}) = \mathrm{VAR}(\hat{\tau}_i) \text{ for all } i. \qquad \text{(C-13b)}$$

Thus, Equation (C-13a) implies that under optimal signal processing, the time delay estimation between any two sensors improves when the number of sensors is increased.

We shall establish Equations (C-13a) and (C-13b) through the following intermediate steps:

(1) Let τ_i, τ_j be any two elements of the incremental time delay vector, then

$$\mathrm{tr}\left(\frac{\partial \Phi}{\partial \tau_i} \frac{\partial \Phi}{\partial \tau_j} \right) = \begin{cases} -2\ i\ (M-j) \ ; \ j \geq i \\ -2\ j\ (M-i) \ ; \ i \geq j \end{cases} \ . \qquad \text{(C-14)}$$

(2)

$$\mathrm{VAR}(\hat{\tau}_i) \geq [J^{-1}]_{ii}$$

$$\geq \left[M \sum_{k=1}^{B} \omega_k^2 \left(\frac{S_k^2/N_k^2}{1 + M\ S_k/N_k} \right) \right]^{-1} \ ; \text{ for } i = 1, 2, \ldots, M-1 \ . \qquad \text{(C-15)}$$

(3) Define $D_{mn} = D_m - D_n$, and $\tau_i = D_{i+1} - D_i$; then

$$\mathrm{VAR}(\hat{D}_{mn}) = \mathrm{VAR}(\hat{\tau}_i) \text{ for all m, n, and i.} \tag{C-16}$$

From Equation (C-8), the mn element of the matrix Φ is given by

$$\Phi^{mn} = \{D_{mn}\} . \tag{C-17}$$

Therefore, the matrix Φ can be written as

$$\Phi = \begin{bmatrix} 0 & -\tau_1 & -(\tau_1+\tau_2) & \cdots & -\sum_{i=1}^{M-1} \tau_i \\ \tau_1 & 0 & -\tau_2 & \cdots & -\sum_{i=1}^{M-2} \tau_i \\ (\tau_1+\tau_2) & \tau_2 & 0 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & -\tau_{M-1} \\ \sum_{i=1}^{M-1} \tau_i & \sum_{i=1}^{M-2} \tau_i & \cdots & \tau_{M-1} & 0 \end{bmatrix} . \tag{C-18}$$

Note that Φ is a skew symmetric matrix (i.e., $\Phi = -\Phi^T$); all elements in the main diagonal are zeros; the incremental time delays occupy the first diagonal; and that any elements above the first diagonal can be obtained by summing the elements to the left and below.

Now it can be verified that

$$\frac{\partial \Phi}{\partial \tau_i} = \left[\begin{array}{c|c} 0_{i x i} & -1_{i x (M-i)} \\ \hline 1_{(M-i) x i} & 0_{(M-i) x (M-i)} \end{array}\right] \tag{C-19}$$

where 0_{ixi} denotes an i x i matrix of 0's and $1_{(M-i)xi}$ denotes an (M-i)xi matrix of 1's.

Therefore, for $j \geq i$, we have

$$\mathrm{tr}\left(\frac{\partial \Phi}{\partial \tau_i}\frac{\partial \Phi}{\partial \tau_j}\right) = \mathrm{tr}\left\{\left[\begin{array}{c|c} A_{j x j} & -B_{j x (M-j)} \\ \hline B^T & 0_{(M-j) x (M-j)} \end{array}\right]\left[\begin{array}{c|c} 0_{j x j} & -1_{j x (M-j)} \\ \hline 1_{(M-j) x j} & 0_{(M-j) x (M-j)} \end{array}\right]\right\} \tag{C-20a}$$

where

$$A_{j x j} = \left[\begin{array}{c|c} 0_{i x i} & -1_{i x (j-i)} \\ \hline 1_{(j-1) x i} & 0_{(j-i) x (j-i)} \end{array}\right] \tag{C-20b}$$

$$B_{j x (M-j)} = \left[\begin{array}{c} -1_{i x (M-j)} \\ \hline 0_{(j-i) x (M-j)} \end{array}\right] . \tag{C-20c}$$

Thus, using Equations (C-20b) and (C-20c) in Equation (C-20a) and carrying out the necessary matrix operation, one finds

$$\mathrm{tr}\left(\frac{\partial\Phi}{\partial\tau_i}\,\frac{\partial\Phi}{\partial\tau_j}\right) = -2i(m-j) \qquad ; \; j \geq i \tag{C-21a}$$

and by symmetry argument, we obtain

$$\mathrm{tr}\left(\frac{\partial\Phi}{\partial\tau_i}\,\frac{\partial\Phi}{\partial\tau_j}\right) = -2j(m-i) \qquad ; \; i \geq j \; . \tag{C-21b}$$

This proves Equation (C-14).

Next we want to show Equation (C-15).

Combining Equations (C-2), (C-5), (C-12) and (C-14), we obtain

$$J = \left[2 \sum_{k=1}^{B} \omega_k^2 \left(\frac{S_k^2/N_k^2}{1 + M\, S_k/N_k} \right) \right] A \tag{C-22}$$

where A is a matrix given by

$$A = \begin{bmatrix} (M-1) & (M-2) & \cdots & 1 \\ (M-2) & 2(M-2) & \cdots & 2 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 1 & 2 & \cdots & (M-1) \end{bmatrix} = \begin{cases} i(M-j) & ; \; j \geq i \\ j(M-i) & ; \; i \geq j \end{cases} \; . \tag{C-23}$$

Thus the inverse of J is

$$J^{-1} = A^{-1}\left[2\sum_{k=1}^{B}\omega_k^2\left(\frac{S_k^2/N_k^2}{1 + M\ S_k/N_k}\right)\right]^{-1} \tag{C-24a}$$

and therefore the diagonal elements of J^{-1} are

$$[J^{-1}]_{ii} = [A^{-1}]_{ii}\left[2\sum_{k=1}^{B}\omega_k^2\left(\frac{S_k^2/N_k^2}{1 + M\ S_k/N_k}\right)\right]^{-1} ; \qquad i = 1, 2, \ldots, M-1 . \tag{C-24b}$$

By inspection, the matrix A^{-1} is given by the M x M matrix

$$A^{-1} = \frac{1}{M}\begin{bmatrix} 2 & -1 & & & 0 \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & \ddots & \ddots & -1 \\ 0 & & & -1 & 2 \end{bmatrix} \tag{C-25a}$$

or the ij element is given by

$$A_{ij}^{-1} = \begin{cases} \frac{2}{M} ; & i = j \\ -\frac{1}{M} ; & |i-j| = 1 . \\ 0 ; & |i-j| > 1 \end{cases} \tag{C-25b}$$

This can be verified by a direct multiplication to show that $A^{-1}A = I$. Therefore, using Equation (C-25b) in Equations (C-24b) and (C-15) yields the desired result:

$$\mathrm{VAR}\ (\hat{\theta}_i) \geq [J^{-1}]_{ii}$$

$$\geq \left[M \sum_{k=1}^{B} \omega_k^2 \left(\frac{S_k^2/N_k^2}{1 + M\ S_k/N_k} \right) \right]^{-1} . \qquad \text{(C-26)}$$

This proves Equation (C-15).

It is easy to see that

$$\mathrm{COV}\ (\hat{\tau}_i, \hat{\tau}_j) = \begin{cases} -\mathrm{VAR}\ (\hat{\tau}_i)/2 & ; \text{ if } |i-j| = 1 \\ 0 & ; \text{ if } |i-j| > 1 \end{cases} . \qquad \text{(C-27)}$$

Finally writing

$$D_{mn} = \sum_{i=n}^{m-1} \tau_i \qquad \text{(C-28)}$$

as the time delay between sensors n and m, the maximum likelihood estimation (MLE) of D_{mn} is linearly related to the MLE of τ_i. Hence, the variance of $\hat{D}_{mn}$ is given by

$$\mathrm{VAR}\ (\hat{D}_{mn}) = \sum_{i=n}^{m-1} \sum_{j=n}^{m-1} E[\hat{\tau}_i\ \hat{\tau}_j] . \qquad \text{(C-29)}$$

Using Equations (C-26) and (C-27) in (C-19) yields

$$\begin{aligned} \mathrm{VAR}(\hat{D}_{mn}) &= (m - n)\ \mathrm{VAR}(\hat{\tau}_i) - (m - n - 1)\ \mathrm{VAR}(\hat{\tau}_i) \\ &= \mathrm{VAR}(\hat{\tau}_i)\ . \end{aligned} \tag{C-30}$$

Thus the optimum MLE of time delay between any two sensors yields the same Cramer-Rao Lower Bound.

APPENDIX D

FUNCTIONAL RELATIONSHIPS BETWEEN TARGET LOCATION VECTOR AND MEASURED TIME DELAY VECTOR

This appendix develops the two-dimensional mathematical relationships between target location parameters and time delay parameters of a general three-sensor array.

Figure D-1 shows the general array/target geometry. In principle, measurement of time delays between sensors A_1, A_2 and A_2, A_3 provides the necessary set of relations to obtain target range and bearing. Using the Law of Cosine, these relationships can be obtained as

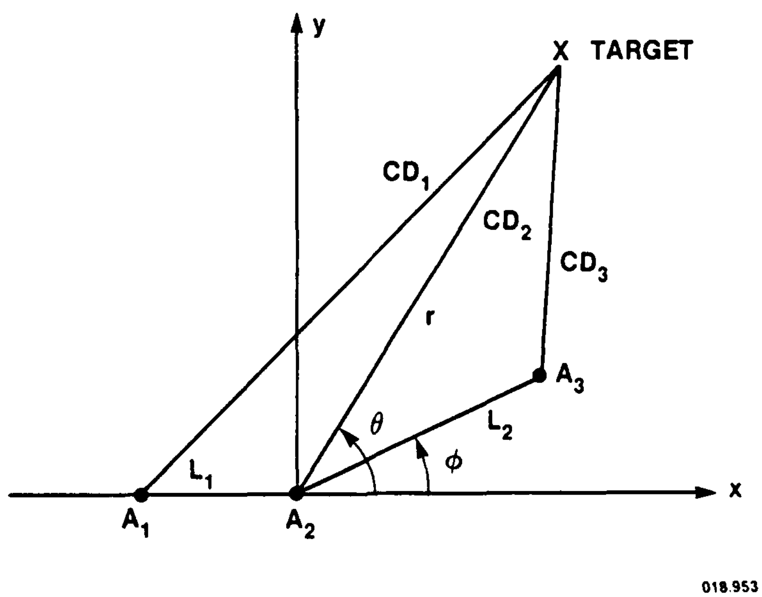


Figure D-1. A General Three-Sensor Passive Ranging Array System

$$\tau_1 = D_2 - D_1$$

$$= \frac{r - \sqrt{r^2 + L_1^2 + 2rL_1 \cos\theta}}{c} \tag{D-1a}$$

and

$$\tau_2 = D_3 - D_2$$

$$= \frac{\sqrt{r^2 + L_2^2 - 2rL_2 \cos(\theta - \phi)} - r}{c} \tag{D-1b}$$

where D_i, i = 1, 2 and 3 are the propagation time delays and ϕ is the offset angle of one sensor with respect to the baseline formed by the remaining two sensors.

For the case where $r >> L_1$, and $r >> L_2$, Equations (D-1a) and (D-1b) become

$$\tau_1 = \frac{r}{c}\left\{1 - \left[1 + \left(\frac{L_1}{r}\right)^2 + 2\left(\frac{L_1}{r}\right)\cos\theta\right]^{1/2}\right\}$$

$$\approx \frac{r}{c}\left\{1 - \left[1 + \frac{1}{2}\left(\frac{L_1}{r}\right)^2 + \left(\frac{L_1}{r}\right)\cos\theta - \frac{1}{2}\left(\frac{L_1}{r}\right)^2 \cos^2\theta\right]\right\}$$

$$\approx -\frac{L_1}{c}\cos\theta - \frac{L_1^2 \sin^2\theta}{2rc} \tag{D-2a}$$

and similarly

$$\tau_2 \approx -\frac{L_2}{c}\cos\theta + \frac{L_2^2 \sin^2\theta}{2rc} \quad . \tag{D-2b}$$

However, the exact expressions for the location parameters as a function of time delays can be found using Equations (D-1a) and (D-1b) as follows.

After rearranging, squaring and simplifying, Equations (D-1a) and (D-1b) reduce to

$$\begin{cases} (c\tau_1)^2 - L_1^{\,2} = 2r(c\tau_1 + L_1 \cos\theta) & \text{(D-3a)} \\ (c\tau_2)^2 - L_2^{\,1} = -2r(c\tau_2 + L_2 \cos(\theta - \phi)\ . & \text{(D-3b)} \end{cases}$$

We first solve for θ, the bearing and then solve for r, the range. Dividing Equations (D-3a) and (D-3b) yields

$$\frac{c\tau_1 + L_1 \cos\theta}{c\tau_2 + L_2 \cos(\theta - \phi)} = -\frac{(c\tau_1)^2 - L_1^2}{(c\tau_2)^2 - L_2^2} = -k \tag{D-4a}$$

so one can write

$$L_1 \cos\theta + KL_2 \cos(\theta - \phi) = -(c\tau_1 + Kc\tau_2) \tag{D-4b}$$

and from which one obtains

$$\theta = \xi + \cos^{-1}\left(-\frac{c\tau_1 + Kc\tau_2}{A}\right) \tag{D-5a}$$

where

$$A \sin\xi = KL_2 \sin\phi$$

$$A \cos\xi = L_1 + KL_2 \cos\phi\ . \tag{D-5b}$$

Therefore, from Equation (D-5a), one obtains

$$\cos \theta = \cos \left[\xi + \cos^{-1} \left(- \frac{c\tau_1 + Kc\tau_2}{A} \right) \right]$$

$$= \frac{(c\tau_1 + Kc\tau_2)\ (L_1 + KL_2 \cos \phi)}{A^2} - \frac{\sqrt{A^2 - (c\tau_1 + Kc\tau_2)^2}}{A^2}\ KL_2 \sin \phi \tag{D-5c}$$

where

$$A^2 = (L_1 + KL_2 \cos \phi)^2 + (KL_2 \sin \phi)^2$$

$$= (L_1 + KL_2)^2\ (1 - \alpha) \tag{D-5d}$$

where

$$\alpha = \frac{2KL_1L_2\ (1 - \cos \phi)}{(L_1 + KL_2)^2}\ . \tag{D-5e}$$

Now substituting Equation (D-5c) into (D-3a), one obtains the range solution as

$$r = \frac{(c\tau_1)^2 - L_1^2}{2(c\tau_1 + L_1 \cos \theta)}$$

$$= \frac{[(c\tau_1)^2 - L_1^2](L_1 + KL_2)\ (1 - \alpha)}{2\{c\tau_1(L_1 + KL_2)(1 - \alpha) - L_1(c\tau_1 + Kc\tau_2)(1 - \beta) - \sqrt{1 - \alpha - \gamma^2}\ KL_1L_2 \sin \phi\}}$$

where

$$\beta = \frac{KL_2(1 - \cos\phi)}{L_1 + KL_2} \tag{D-6b}$$

$$\gamma = \frac{c\tau_1 + Kc\tau_2}{L_1 + KL_2} \quad . \tag{D-6c}$$

From the relation (D-4a)

$$K = \left(\frac{L_1}{L_2}\right)^2 \frac{1 - \left(\frac{c\tau_1}{L_1}\right)^2}{1 - \left(\frac{c\tau_2}{L_2}\right)^2} \tag{D-6d}$$

Equation (D-6) can be simplified to

$$r = \frac{L_1 \left[1 - \left(\frac{c\tau_1}{L_1}\right)^2 + L_2 \;\; 1 - \left(\frac{c\tau_2}{L_2}\right)^2\right]}{2\left[\left(\frac{c\tau_2}{L_2}\right) - \left(\frac{c\tau_1}{L_1}\right) + \Gamma(\phi)\right]} (1 - \alpha) \tag{D-7a}$$

where

$$\Gamma(\phi) = \frac{c\tau_1}{KL_2} (\alpha - \beta) + \left[\left(\frac{c\tau_1}{L_1}\right)\alpha - \left(\frac{c\tau_2}{L_2}\right)\beta\right] + \sqrt{1 - \alpha - \gamma^2} \;\sin\phi \tag{D-7b}$$

is a function of the offset angle ϕ.

Now further algebraic manipulation shows that

$$\frac{c\tau_1}{KL_2}(\alpha - \beta) = \frac{c\tau_1}{KL_2}\left[\frac{2KL_1L_2(1 - \cos\phi)}{(L_1 + KL_2)^2} - \frac{KL_2\,(1 - \cos\phi)}{L_1 + KL_2}\right]$$

$$= \left(\frac{L_1 - KL_2}{L_1 + KL_2}\right)(1 - \cos\phi)\; c\tau_1 \tag{D-7c}$$

and

$$\left(\frac{c\tau_1}{L_1}\right)\alpha - \left(\frac{c\tau_2}{L_2}\right)\beta = \frac{c\tau_1\; 2KL_2(1 - \cos\phi)}{(L_1 + KL_2)^2} - \frac{c\tau_2\; K(1 - \cos\phi)}{(L_1 + KL_2)}$$

$$= \frac{K(1 - \cos\phi)}{(L_1 + KL_2)^2}\,[2\, L_2\, c\tau_1 - c\tau_2(L_1 + KL_2)]\;. \tag{D-7d}$$

Combining Equations (D-7c) and (D-7d) yields

$$\left(\frac{c\tau_1}{L_1}\right)(\alpha - \beta) + \left(\frac{c\tau_1}{L_1}\right)\alpha - \left(\frac{c\tau_2}{L_2}\right)\beta$$

$$= \frac{1 - \cos\phi}{(L_1 + KL_2)^2}\,[c\tau_1(L_1 - KL_2) + 2KL_2\, c\tau_1 - Kc\tau_2(L_1 + KL_2)]$$

$$= \frac{1 - \cos\phi}{(L_1 + KL_2)^2}\;(L_1 + KL_2)\,(c\tau_1 - Kc\tau_2)$$

$$= \frac{c\tau_1 - Kc\tau_2}{L_1 + KL_2}\;(1 - \cos\phi)\;. \tag{D-7e}$$

Thus Equation (D-7b) becomes

$$\Gamma(\phi) = \frac{c\tau_1 - Kc\tau_2}{L_1 + KL_2}(1 - \cos\phi) + \sqrt{1 - \alpha - \gamma^2}\,\sin\phi\,. \tag{D-7f}$$

Note for a colinear array system (i.e., $\phi = 0$), we have $\alpha = 0$, $\Gamma(0) = 0$, and the range and bearing equations are given by

$$r = \frac{L_1\left[1 - \left(\frac{c\tau_1}{L_1}\right)^2\right] + L_2\left[1 - \left(\frac{c\tau_2}{L_2}\right)^2\right]}{2\left[\left(\frac{c\tau_2}{L_2}\right) - \left(\frac{c\tau_1}{L_1}\right)\right]} \tag{D-8a}$$

and

$$\theta = \cos^{-1}\left(- \frac{c\tau_1 + Kc\tau_2}{L_1 + KL_2}\right). \tag{D-8b}$$

In addition for $L_1 = L_2 = L$ and $r >> L_1$, using Equations (D-2a) and (D-2b), it can be shown that

$$\theta = \cos^{-1}\left[- \frac{c(\tau_1 + \tau_2)}{2L}\right] \tag{D-9a}$$

$$r = \frac{L^2 \sin^2\theta}{c(\tau_2 - \tau_1)}\,. \tag{D-9b}$$

If θ is defined w.r.t. the broadside direction, Equations (D-9a) and (D-9b) can be written as

$$\theta = -\sin^{-1}\left[\frac{c}{2L}\left(\tau_1 + \tau_2\right)\right] \tag{D-10a}$$

$$r = \frac{L^2}{c}\frac{\cos^2\theta}{(\tau_2 - \tau_1)} \tag{D-10b}$$

APPENDIX E

STATISTICAL CORRELATION BETWEEN TIME DELAY ESTIMATES WITH COMMON INPUT CHANNEL

If time delay estimates are obtained from two Generalized Cross-Correlators (GCC) whose inputs contain a common noise channel (Figure E-1), then the resulting time delay estimates are correlated. This appendix calculates the resulting correlation from a frequency domain approach.

Let the frequency domain (Fourier) representation of a T-second observation of waveforms from sensor array A_1, A_2, and A_3 with signal propagation delays D_1, D_2 and D_3, respectively, be written as:

$$\alpha_{1k} = \beta_k\, e^{j\omega_k D_1} + n_{1k} \tag{E-1a}$$

$$\alpha_{2k} = \beta_k\, e^{j\omega_k D_2} + n_{2k} \tag{E-1b}$$

$$\alpha_{3k} = \beta_k\, e^{j\omega_k D_3} + n_{3k} \quad ; \quad k = 1, 2, \ldots, B \tag{E-1c}$$

with $S_k = E(\beta_k \beta_k^*)$ and $N_{ik} = E(n_{ik} n_{ik}^*)$ for all i as the discrete signal and noise power spectra, respectively, at frequency $\omega_k = 2\pi k/T$. For convenience we assume that interarray noise processes are uncorrelated. The time delay $\tau_1 = D_2 - D_1$ and $\tau_2 = D_3 - D_2$ can be obtained by seeking the null of the GCC functions (Equation (3.5.1-14c)).

$$f_1(\tau_1) = \sum_{k=-B}^{B} j\omega_k\, h_{1k}\, h_{2k}^*\, \alpha_{1k}\, \alpha_{2k}^*\, e^{j\omega_k \tau_1}$$

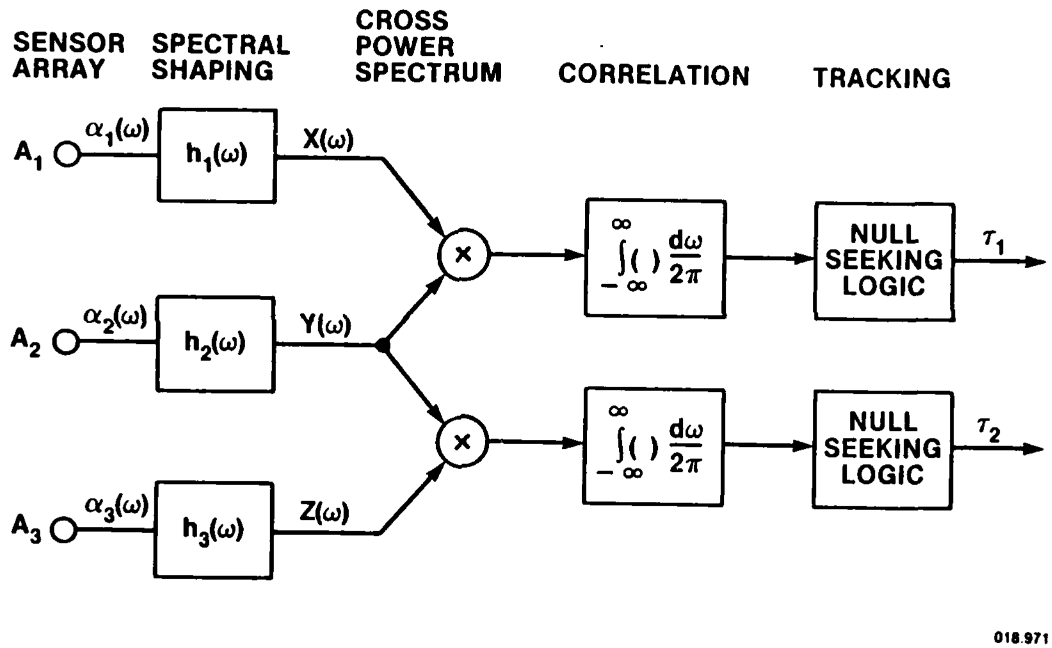


Figure E-1. Generalized Cross Correlator Pair with Common Input Channel

and

$$f_2(\tau_2) = \sum_{k=-B}^{B} j\omega_k \, h_{2k}^{*} \, h_{3k} \, \alpha_{2k} \, \alpha_{3k}^{*} \, e^{j\omega_k \tau_2} \tag{E-2b}$$

where B is the uppermost frequency of either the signal or the noise and h_{ik} is the frequency shaping filter whose frequency response is

$$|h_{ik}|^2 = \frac{S_k/N_{ik}^2}{1 + 2\, S_k/N_{ik}} \; . \tag{E-2c}$$

Let $\hat{\tau}_1$ and $\hat{\tau}_2$ be the estimated time delay, then the quantity of interest is the covariance between $\hat{\tau}_1$ and $\hat{\tau}_2$; i.e., $\mathrm{COV}(\hat{\tau}_1, \hat{\tau}_2)$. The Taylor series expansion of $f_1(\tau_1)$ and $f_2(\tau_2)$ about the true time delay τ_1^0 and τ_2^0 yields

$$f_1(\tau_1) = f_1(\tau_1^0) + \left.\frac{\partial f_1(\tau_1)}{\partial \tau_1}\right|_{\tau_1 = \tau_1^0} (\tau_1 - \tau_1^0) + \ldots \tag{E-3a}$$

and

$$f_2(\tau_2) = f_2(\tau_2^0) + \left.\frac{\partial f_2(\tau_2)}{\partial \tau_2}\right|_{\tau_2 = \tau_2^0} (\tau_2 - \tau_2^0) + \ldots \; . \tag{E-3b}$$

Neglecting the higher order terms (small error assumption) and setting $f_1(\tau_1) = f_2(\tau_2) = 0$ in Equations (E-3a) and (E-3b), one obtains

$$-f_1(\tau_1^0) = \frac{\partial f_1(\tau_1^0)}{\partial \tau_1} (\tau_1 - \tau_1^0) \tag{E-4a}$$

$$-f_2(\tau_2^0) = \frac{\partial f_2(\tau_2^0)}{\partial \tau_2} (\tau_2 - \tau_2^0) \ . \tag{E-4b}$$

We shall make the following assumptions as a result of a long observation time process: (1) the expected value of the derivative is equal to the derivative of the expected value, and (2) the derivative is uncorrelated with time delay. Then from Equations (E-4a) and (E-4b), one obtains

$$\overline{-f_1^0} = \overline{\frac{\partial f_1^0}{\partial \tau_1}}\, \overline{(\tau_1 - \tau_1^0)} \tag{E-5a}$$

$$\overline{-f_2^0} = \overline{\frac{\partial f_2^0}{\partial \tau_2}}\, \overline{(\tau_2 - \tau_2^0)} \tag{E-5b}$$

where for simplicity we denote the expected values of $f_1(\tau_1^0)$ and $f_2(\tau_2^0)$ by $\overline{f_1^0}$ and $\overline{f_2^0}$. Thus, the product of the means is

$$\overline{f_1^0}\ \overline{f_2^0} = \overline{\frac{\partial f_1^0}{\partial \tau_1}}\ \overline{\frac{\partial f_2^0}{\partial \tau_2}}\ \overline{(\tau_1 - \tau_1^0)}\ \overline{(\tau_2 - \tau_2^0)} \ . \tag{E-6}$$

On the other hand, the mean of the product is

$$\overline{f_1^0 \; f_2^0} = \overline{\frac{\partial f_1^0}{\partial \tau_1}} \; \overline{\frac{\partial f_2^0}{\partial \tau_2}} \; \overline{(\tau_1 - \tau_1^0) \; (\tau_2 - \tau_2^0)} \quad . \tag{E-7}$$

Therefore, the covariance is

$$\mathrm{COV}(\hat{\tau}_1,\hat{\tau}_2) = \overline{(\tau_1 - \tau_1^0)(\tau_2 - \tau_2^0)} \; - \overline{(\tau_1 - \tau_1^0)} \; \overline{(\tau_2 - \tau_2^0)}$$

$$= \frac{\overline{f_1^0 \; f_2^0} - \overline{f_1^0} \; \overline{f_2^0}}{\left(\overline{\dfrac{\partial f_1^0}{\partial \tau_1}} \; \overline{\dfrac{\partial f_2^0}{\partial \tau_2}}\right)} \quad . \tag{E-8}$$

From Equations (E-2a) and (E-2b), the quantities in Equation (E-8) can be evaluated as follows.

$$\overline{\frac{\partial f_1^0}{\partial \tau_1}} = \sum_{k=-B}^{B} (j\omega_k)^2 \; h_{1k} \; h_{2k}^* \; S_k \tag{E-9a}$$

$$\overline{\frac{\partial f_2^0}{\partial \tau_2}} = \sum_{k=-B}^{B} (j\omega_k)^2 \; h_{2k} \; h_{3k}^* \; S_k \tag{E-9b}$$

$$\overline{f_1^o} = \sum_{k=-B}^{B} j\omega_k \; h_{1k} \; h_{2k}^* \; S_k \tag{E-9c}$$

$$\overline{f_2^o} = \sum_{k=-B}^{B} j\omega_k \; h_{2k} \; h_{3k}^* \; S_k \tag{E-9d}$$

and

$$\overline{f_1^o \; f_2^o} = \sum_{k=-B}^{B} \sum_{\ell=-B}^{B} (j\omega_k)(j\omega\ell) \; h_{1k} \; h_{2k}^* \; h_{2\ell} \; h_{3\ell}^* \; \overline{\alpha_{1k} \; \alpha_{2k}^* \; \alpha_{2\ell} \; \alpha_{3\ell}^*} \; e^{j(\omega_k \tau_1^o + \omega_\ell \tau_2^o)} \; . \tag{E-10}$$

But

$$\overline{\alpha_{1k} \; \alpha_{2k}^* \; \alpha_{2\ell} \; \alpha_{3\ell}^*} = \overline{\alpha_{1k} \; \alpha_{2k}^*} \; \overline{\alpha_{2\ell} \; \alpha_{3\ell}^*} + \overline{\alpha_{1k} \; \alpha_{2\ell}} \; \overline{\alpha_{2k}^* \; \alpha_{3\ell}^*} + \overline{\alpha_{1k} \; \alpha_{3\ell}^*} \; \overline{\alpha_{2k}^* \; \alpha_{2\ell}}$$

$$= S_k^2 \; e^{-j(\omega_k \tau_1^o + \omega_\ell \tau_2^o)} + S_k^2 \; e^{-j\omega_k (\tau_1^o + \tau_2^o)} \; \delta_{k+\ell}$$

$$+ S_k(S_k + N_{2k}) \; e^{-j\omega_k (\tau_1^o + \tau_2^o)} \; \delta_{k-\ell} \; , \tag{E-11}$$

where $\delta_{k-\ell}$ is the Kronecka delta defined by $\delta_{k-\ell} = 1$ if $k = \ell$; and $\delta_{k-\ell} = 0$ if $k \neq \ell$.

Now substituting Equation (E-11) into Equation (E-10) yields;

$$\overline{f_1^o \, f_2^o} = \sum_{k=-B}^{B} \sum_{\ell=-B}^{B} (j\omega_k)\,(j\omega_\ell)\, h_{1k}\, h_{2k}^*\, h_{2\ell}\, h_{3\ell}^*\, S_k^2$$

$$+ \sum_{k=-B}^{B} \omega_k^2\, h_{1k}\, h_{3k}^*\, |h_{2k}|^2\, S_k^2$$

$$- \sum_{k=-B}^{B} \omega_k^2\, h_{1k}\, h_{3k}^*\, |h_{2k}|^2\, S_k(S_k + N_{2k})$$

$$= \left(\sum_{k=-B}^{B} j\omega_k\, h_{1k}\, h_{2k}^*\, S_k \right) \left(\sum_{k=-B}^{B} j\omega_k\, h_{2k}\, h_{3k}^*\, S_k \right)$$

$$- \sum_{k=-B}^{B} \omega_k^2\, h_{1k}\, h_{3k}^*\, |h_{2k}|^2\, S_k\, N_{2k}$$

$$= \overline{f_1^o}\;\overline{f_2^o} - \sum_{k=-B}^{B} \omega_k^2\, h_{1k}\, h_{3k}^*\, |h_{2k}|^2\, S_k\, N_{2k} \quad . \qquad \text{(E-12)}$$

Finally, using Equations (E-9a) through (E-12) in Equation (E-8) yields

$$\mathrm{COV}(\hat{\tau}_1, \hat{\tau}_2) = \frac{-\sum_{k=-B}^{B} \omega_k^2\, h_{1k}\, h_{3k}^*\, |h_{2k}|^2\, S_k\, N_{2k}}{\left(\sum_{k=-B}^{B} \omega_k^2\, h_{1k}\, h_{2k}^*\, S_k \right)\left(\sum_{k=-B}^{B} \omega_k^2\, h_{2k}\, h_{3k}^*\, S_k \right)} \quad . \qquad \text{(E-13)}$$

Equation (E-13) can also be expressed in continuous frequency domain as

$$\mathrm{COV}(\hat{\tau}_1,\hat{\tau}_2) = -\frac{2\pi}{T}\,\frac{\int_0^B \omega^2\, h_1(\omega)\, h_3^*(\omega)\, |h_2(\omega)|^2\, S(\omega)\, N_2(\omega)\, d\omega}{\int_0^B \omega^2\, h_1\, h_2^*(\omega)\, S(\omega)\, d\omega \int_0^B \omega^2\, h_2(\omega)\, h_3^*(\omega)\, S(\omega)\, d\omega}\ . \tag{E-14}$$

Note that if either the definition of τ_1 or τ_2 is reversed, the sign in Equation (E-13) and (E-14) is also reversed.

Finally, it can also be shown similarly that the variances of the estimates are given by:

$$\mathrm{VAR}(\hat{\tau}_1) = \frac{\overline{f_1^{o\,2}} - \overline{f_1^o}^{\,2}}{\left(\overline{\dfrac{\partial f_1^o}{\partial \tau_1}}\right)^2}$$

$$= \frac{\sum_{k=-B}^{B} \omega_k^2\, |h_{1k}|^2\, |h_{2k}|^2\, [S_k(N_{1k} + N_{2k}) + N_{1k}\, N_{2k}]}{\left(\sum_{k=-B}^{B} \omega_k^2\, h_{1k}\, h_{2k}^*\, S_k\right)^2}$$

$$= \frac{2\pi}{T} \frac{\int_0^B \omega^2 \, |h_1(\omega)|^2 \, |h_2(\omega)|^2 \, [S(N_1 + N_2) + N_1 N_2] \, d\omega}{\left(\int_0^B \omega^2 \, h_1(\omega) \, h_2^*(\omega) \, S(\omega) \, d\omega\right)^2} \quad . \qquad \text{(E-15)}$$

Similarly, one obtains

$$\mathrm{VAR}(\hat{\tau}_2) = \frac{2\pi}{T} \frac{\int_0^B \omega^2 \, |h_2(\omega)|^2 \, |h_3(\omega)|^2 \, [S(N_2 + N_3) + N_2 N_3] \, d\omega}{\left(\int_0^B \omega^2 \, h_2(\omega) \, h_3^*(\omega) \, S(\omega) \, d\omega\right)^2} \quad . \qquad \text{(E-16)}$$

A substantial simplification can be obtained for the case of bandlimited flat signal and noise power spectra.

Let

$$S(\omega) = \begin{cases} S \text{ if } \omega_1 \leq \omega \leq \omega_2 \\ 0 \quad \text{otherwise} \end{cases}$$

$$N_i(\omega) = \begin{cases} N \text{ if } \omega_1 \leq \omega \leq \omega_2 \text{ and for all } i \\ 0 \quad \text{otherwise} \end{cases}$$

be the signal and the noise power spectra and assume identical frequency spectral filters for all channels, then the covariance expression of Equation (E-14) reduces to

$$\mathrm{COV}(\hat{\tau}_1,\hat{\tau}_2) = -\frac{2\pi}{T}\left(\frac{S}{N}\int_0^B \omega^2\, d\omega\right)^{-1}$$

$$= -\frac{2\pi}{T}\left[\left(\frac{2}{3}\right)\left(\frac{S}{N}\right)(\omega_2^3 - \omega_1^3)\right]^{-1} \tag{E-17}$$

and the variances reduce to

$$\mathrm{VAR}(\hat{\tau}_1) = \mathrm{VAR}(\hat{\tau}_2)$$

$$= \frac{2\pi}{T}\left[\frac{(S/N)^2}{1 + 2\, S/N}\int_0^B \omega^2\, d\omega\right]^{-1}$$

$$= \frac{2\pi}{T}\left[\frac{2\,(S/N)^2}{3(1 + 2\, S/N)}(\omega_2^3 - \omega_1^3)\right]^{-1} . \tag{E-18}$$

Comparison between Equations (E-18) and (E-17) shows that

$$\mathrm{COV}(\hat{\tau}_1,\hat{\tau}_2) = -\frac{S/N}{1 + 2\, S/N}\,\mathrm{VAR}(\hat{\tau}_1) . \tag{E-19}$$

APPENDIX F
DERIVATION OF EQUATION (3.5.2-7a)

From Equation (3.5.2-7a) we have

$$\frac{\partial \Lambda}{\partial r}(r,\theta) = \sum_{k=1}^{B} j\omega_k \; |h_k|^2 \; \underline{\alpha}_k^* \; V_k \; \frac{\partial \Phi}{\partial r} \; V_k^* \; \underline{\alpha}_k \quad . \tag{F-1}$$

Using the relation

$$\frac{\partial \Phi}{\partial r} = \frac{\partial \Phi}{\partial \tau_1} \frac{\partial \tau_1}{\partial r} + \frac{\partial \Phi}{\partial \tau_2} \frac{\partial \tau_2}{\partial r} \tag{F-2}$$

and Equation (3.5.1-10a-c)

$$\frac{\partial}{\partial \tau_i} (V_k \; 1_M \; V_k^*) = j\omega_k \; V_k \; \frac{\partial \Phi}{\partial \tau_i} \; V_k^* \tag{F-3}$$

in Equation (F-1) yields

$$\frac{\partial \Lambda(r,\theta)}{\partial r} = \left\{ \sum_{k=1}^{B} j\omega_k \; |h_k|^2 \; \underline{\alpha}_k^* \; V_k \; \frac{\partial \Phi}{\partial \tau_1} \; V_k^* \; \underline{\alpha}_k \right\} \frac{\partial \tau}{\partial r}$$

$$+ \left\{ \sum_{k=1}^{B} j\omega_k \; |h_k|^2 \; \underline{\alpha}_k^* \; V_k \; \frac{\partial \Phi}{\partial \tau_2} \; V_k^* \; \underline{\alpha}_k \right\} \frac{\partial \tau_2}{\partial r}$$

$$= \frac{\partial}{\partial \tau_1} \left\{ \sum_{k=1}^{B} |h_k|^2 \; \underline{\alpha}_k^* \; V_k \; 1_M \; V_k^* \; \underline{\alpha}_k \right\} \frac{\partial \tau_1}{\partial r}$$

$$+ \frac{\partial}{\partial \tau_2} \left\{ \sum_{k=1}^{B} |h_k|^2 \underset{\sim}{\alpha}_k^* V_k 1_M V_k^* \underset{\sim}{\alpha}_k \right\} \frac{\partial \tau_2}{\partial r}$$

$$= \frac{\partial}{\partial r} \sum_{k=1}^{B} |h_k|^2 \underset{\sim}{\alpha}_k^* V_k 1_M V_k^* \underset{\sim}{\alpha}_k \quad . \tag{F-4}$$

But

$$V_k 1_M V_k^* = \begin{bmatrix} e^{-j\omega_k \tau_1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-j\omega_k \tau_2} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} e^{j\omega_k \tau_1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{j\omega_k \tau_2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & e^{-j\omega_k \tau_1} & e^{-j\omega_k (\tau_1+\tau_2)} \\ e^{j\omega_k \tau_1} & 1 & e^{-j\omega_k \tau_2} \\ e^{j\omega_k (\tau_1+\tau_2)} & e^{j\omega_k \tau_2} & 1 \end{bmatrix} \tag{F-5}$$

and

$$\underset{\sim}{\alpha}_k^* V_k 1_M V_k^* \underset{\sim}{\alpha}_k = \left\{ |\alpha_{1k}|^2 + |\alpha_{2k}|^2 + |\alpha_{3k}|^2 \right\}$$

$$+ \left\{ (\alpha_{2k}^* \alpha_{1k} e^{j\omega_k \tau_1}) + (\alpha_{2k}^* \alpha_{1k} e^{j\omega_k \tau_1})^* \right\}$$

$$+ \left\{ (\alpha_{3k}^{*} \alpha_{2k} e^{j\omega_k \tau_2}) + (\alpha_{3k}^{*} \alpha_{2k} e^{j\omega_k \tau_2})^{*} \right\}$$

$$+ \left\{ (\alpha_{3k}^{*} \alpha_{1k} e^{j\omega_k (\tau_1 + \tau_2)}) + (\alpha_{3k}^{*} \alpha_{1k} e^{j\omega_k (\tau_1 + \tau_2)})^{*} \right\} . \tag{F-6}$$

Now using the fact that the first term in Equation (F-6) is not parameter-dependent and that quantities in the parentheses are conjugate symmetric, Equation (F-4) can be rewritten as

$$\frac{\partial \Lambda(r,\theta)}{\partial r} = \frac{\partial}{\partial r} \sum_{k=-B}^{B} |h_k|^2 \left\{ \alpha_{2k}^{*} \alpha_{1k} e^{j\omega_k \tau_1} + \alpha_{3k}^{*} \alpha_{2k} e^{j\omega_k \tau_2} + \alpha_{3k}^{*} \alpha_{1k} e^{j\omega_k (\tau_1 + \tau_2)} \right\}$$

$$= \frac{T}{2\pi} \frac{\partial}{\partial r} \int_{-\infty}^{\infty} |h(\omega)|^2 \, G(\omega;\, \tau_1,\, \tau_2) \, d\omega \tag{F-7}$$

where

$$G(\omega;\, \tau_1,\, \tau_2) = T \left[(\alpha_{2k}^{*} \alpha_{1k}) e^{j\omega_k \tau_1} + (\alpha_{3k}^{*} \alpha_{2k}) e^{j\omega_k \tau_2} + (\alpha_{3k}^{*} \alpha_{1k}^{*}) e^{j\omega_k (\tau_1 + \tau_2)} \right] . \tag{F-8}$$

APPENDIX G

CALCULATION OF LOCALIZATION UNCERTAINTY FROM THREE-SENSOR ARRAYS

This appendix calculates the localization uncertainty based on a two inter-array time delay measurement.

From Equations (D-10a) and D-10b), the approximate ($r >> L$) time delays to range and bearing relation are given by

$$\theta = -\sin^{-1} \left\{\frac{c}{2L} (\tau_1 + \tau_2)\right\} \tag{G-1}$$

$$r = \frac{L^2}{c} \frac{\cos^2 \theta}{(\tau_2 - \tau_1)} \tag{G-2}$$

where c is the speed of sound, and L is the inter-array separation.

Letting $\underline{Y} = (\theta, r)^T$, $\underline{X} = (\tau_1, \tau_2)$, Equations (G-1) and (G-2) can be written as

$$\underline{Y} = \underline{f}(\underline{X})$$

where $\underline{f}(X) = [f_1(\underline{X})\ f_2(\underline{X})]^T$ is the vector function defined according to Equations (G-1) and (G-2). Let $\underline{Y}^*$ and $\underline{X}^*$ be the nominal values, $\delta\underline{Y}$ and $\delta\underline{X}$ be the deviations from the nominal; then a first-order expansion about the nominal (R, B) yields

$$\delta\underline{Y} = \frac{\partial f^*}{\delta\underline{X}} \delta\underline{X} \tag{G-3}$$

where it is defined that

$$\frac{\partial \underline{f}^*}{\partial \underline{X}} = \begin{bmatrix} \frac{\partial f_1^*}{\partial \theta} & \frac{\partial f_1^*}{\partial r} \\ \frac{\partial f_2^*}{\partial \theta} & \frac{\partial f_2^*}{\partial r} \end{bmatrix}$$

$$= \begin{bmatrix} a & a \\ b & -b \end{bmatrix} \tag{G-5a}$$

where

$$a = -\frac{c}{2L \cos B} \tag{G-5b}$$

$$b = \frac{R^2 c}{L^2 \cos^2 B} \tag{G-5c}$$

since R >> L.

Now post-multipling Equation (G-4) by its transpose and taking the expected value yields the desired covariance relations:

$$\mathrm{COV}(\underline{Y}) = \left(\frac{\partial \underline{f}^*}{\partial \underline{X}}\right) \mathrm{COV}(\underline{X}) \left(\frac{\partial \underline{f}^*}{\partial \underline{X}}\right)^T . \tag{G-6}$$

Let

$$\mathrm{COV}(\underline{X}) = \begin{bmatrix} \sigma 2_{11} & \sigma 2_{12} \\ \sigma 2_{12} & \sigma 2_{11} \end{bmatrix} \tag{G-7}$$

then

$$\mathrm{COV}(\underline{Y}) = \begin{bmatrix} a & a \\ \\ b & -b \end{bmatrix} \begin{bmatrix} \sigma 2 \atop 11 & \sigma 2 \atop 12 \\ \\ \sigma 2 \atop 12 & \sigma 2 \atop 11 \end{bmatrix} \begin{bmatrix} a & b \\ \\ a & -b \end{bmatrix}$$

$$= \begin{bmatrix} 2(\sigma_{11}^2 + \sigma_{12}^2)a^2 & 0 \\ \\ 0 & 2(\sigma_{11}^2 - \sigma_{12}^2)b^2 \end{bmatrix} \quad \text{(G-8)}$$

and the determinant of $\mathrm{COV}(\underline{Y})$ is

$$|\mathrm{COV}(\underline{Y})| = 4\ a^2\ b^2(\sigma_{11}^4 - \sigma_{12}^4) \quad . \quad \text{(G-9)}$$

The area of the one-sigma error ellipse is

$$S = \pi\ |\mathrm{COV}(\underline{Y})|^{\frac{1}{2}}$$

$$= -2\ ab\ \pi \sqrt{\sigma_{11}^4 - \sigma_{12}^4}$$

$$= \frac{\pi(Rc)^2}{L^3\ \cos^3\ B} \sqrt{\sigma_{11}^4 - \sigma_{12}^4}\ . \quad \text{(G-10)}$$

The time delay covariance matrix for the optimum approach is (see Equation (3.5.2-16d))

$$P_o = \begin{bmatrix} 1 & -\frac{1}{2} \\ \\ -\frac{1}{2} & 1 \end{bmatrix} \mathrm{VAR}(\hat{\tau}_1) \quad \text{(G-11)}$$

and the conventional is

$$P_c = \begin{bmatrix} 1 & -\frac{S/N}{1 + 2\ S/N} \\ -\frac{S/N}{1 + 2\ S/N} & 1 \end{bmatrix} VAR_c(\hat{\tau}_1) \quad . \tag{G-12}$$

Thus the corresponding area is

$$S_o = \sqrt{\frac{3}{4}} \quad \frac{\pi(Rc)^2}{L^3 \cos^3 B} \; VAR(\hat{\tau}) \tag{G-13}$$

and

$$S_c = \frac{\pi(Rc)^2}{L^3 \cos^3 B} \; \frac{\sqrt{(1 + 3\ S/N)(1 + S/N)}}{(1 + 2\ S/N)} \; VAR_c(\hat{\tau}_1) \tag{G-14}$$

where $VAR(\tau_1)$ and $VAR_c(\tau_1)$ are the estimated time delay variances of the optimum and the conventional processors, respectively.

Finally, from Equations (G-8) and (G-11), the optimum range and bearing variance expressions are given by

$$\begin{aligned} VAR(\hat{B}) &= 2(\sigma_{11}^2 + \sigma_{12}^2)\ a^2 \\ &= \frac{1}{4}\left(\frac{c}{L \cos B}\right)^2 VAR(\hat{\tau}_1) \end{aligned} \tag{G-15}$$

$$\mathrm{VAR}(\hat{R}) = 2(\sigma_{11}^2 - \sigma_{12}^2)\, b^2$$

$$= 3c^2 \left(\frac{R}{L \cos B}\right)^4 \mathrm{VAR}(\hat{\tau}_1) \qquad \text{(G-16)}$$

and from Equations (G-8) and (G-12), the conventional range and bearing variance expressions are:

$$\mathrm{VAR}_c(\hat{B}) = \frac{1}{2}\left(\frac{c}{L \cos B}\right)^2 \left(\frac{1 + S/N}{1 + 2\ S/N}\right) \mathrm{VAR}_c(\hat{\tau}_1) \qquad \text{(G-17a)}$$

$$\mathrm{VAR}_c(\hat{R}) = 2c^2 \left(\frac{R}{L \cos B}\right)^4 \left(\frac{1 + 3\ S/N}{1 + 2\ S/N}\right) \mathrm{VAR}_c(\hat{\tau}_1)\ . \qquad \text{(G-17b)}$$

Using the relation given in Equation (3.5.2-16c),

$$\frac{\mathrm{VAR}(\hat{\tau}_1)}{\mathrm{VAR}_c(\hat{\tau}_1)} = \frac{2}{3}\left(\frac{1 + 3\ S/N}{1 + 2\ S/N}\right),$$

the conventional range and bearing variance equation can also be expressed as:

$$\mathrm{VAR}_c(\hat{B}) = \frac{3}{4}\left(\frac{c}{L \cos B}\right)^2 \left(\frac{1 + S/N}{1 + 3\ S/N}\right) \mathrm{VAR}(\hat{\tau}_1)$$

$$= 3\left(\frac{1 + S/N}{1 + 3\ S/N}\right) \mathrm{VAR}(\hat{B}) \qquad \text{(G-18)}$$

and

$$\mathrm{VAR}_c(\hat{R}) = 3c^2 \left(\frac{R}{L \cos B}\right)^4 \mathrm{VAR}(\hat{\tau}_1)$$

$$= \mathrm{VAR}(\hat{R}) \; . \tag{G-19}$$

Equations (G-18) and (G-19) show that for estimating range, the conventional approach and the optimal approach have identical variance. On the other hand for estimating bearing, the conventional yields a variance which is three times the optimum at low SNR environment.

APPENDIX H

GCC STATISTICAL PERFORMANCE IN THE PRESENCE OF INTERFERENCE

This appendix develops the appropriate expressions for the bias and variance of the time delay estimate from a Generalized Cross-Correlator (GCC) in the presence of interference.

Let the frequency domain representation of a T-second observation of waveforms from two-sensor arrays, A_1 and A_2, in the presence of J targets be written as:

$$\alpha_{1k} = \underline{V}_{1k}\, \underline{\beta}_k + n_{1k} \tag{H-1a}$$

$$\alpha_{2k} = \underline{V}_{2k}\, \underline{\beta}_k + n_{2k}\ ; \qquad k=1,\ 2,\ \ldots,\ B \tag{H-1b}$$

where B is the highest frequency components of the signal or the noise processes; n_{1k} and n_{2k} are the Fourier components of the noise processes at frequency $\omega_k = 2\pi k/T$. In addition, the complex vectors $\underline{\beta}_k$, $\underline{V}_{1k}$ and $\underline{V}_{2k}$ are defined by

$$\underline{V}_{ik} = [e^{j\omega_k D_{i1}},\ e^{j\omega_k D_{i2}},\ .\ .\ .,\ e^{j\omega_k D_{iJ}}];\ i = 1,\ 2 \tag{H-1c}$$

and

$$\underline{\beta}_k = [\beta_{k1},\ \beta_{k2},\ .\ .\ .,\ \beta_{kJ}]^T \tag{H-1d}$$

where D_{ij} is the propagation time delay from target j to sensor i and β_{kj} is the Fourier component of the signal spectrum of target j. For convenience, it is assumed that both the signal and noise processes are zero mean and mutually uncorrelated. Thus, we have the relations for the discrete signal and noise power spectra

$$\overline{\beta_{ki}\ \beta^*_{kj}} = \begin{cases} S_{kj};\ \text{if } i = j \\ 0\ \ ;\ \text{if } i \neq j \end{cases} \tag{H-2a}$$

and

$$\overline{\eta_{ik}\ \eta^*_{jk}} = \begin{cases} N_k;\ \text{if } i = j \\ 0\ ;\ \text{if } i \neq j \end{cases} \tag{H-2b}$$

where $(\overline{\ \ })$ denotes the expected value.

Without loss of generality, let target $j = 1$ be the target of interest, then the best estimate of $\tau_1 = D_{21} - D_{11}$ from a GCC (see Equation (3.5.1-14b)) is to seek the null of the equation:

$$f(\tau) = \sum_{k=-B}^{B} j\omega_k |h_k|^2\ \alpha_{1k}\ \alpha^*_{2k}\ e^{j\omega_k \tau} = 0 \tag{H-3a}$$

where $|h_k|^2$, given by

$$|h_k|^2 = \frac{S_{k1}/N_k^2}{1 + 2\ S_{k1}/N_k}\ , \tag{H-3b}$$

is the optimum spectral shaping filter for a single target environment. Because of the observation noise, Equation (H-3) is a stochastic algebraic equation. On average, however, the mean of the estimate must satisfy the equation

$$\overline{f}(\tau) = \sum_{k=-B}^{B} j\omega_k |h_k|^2\ \overline{\alpha_{1k}\ \alpha^*_{2k}}\ e^{j\omega_k \tau} = 0. \tag{H-4a}$$

But from Equations (H-1a) and (H-1b), one obtains

$$\overline{\alpha_{1k}\ \alpha_{2k}^{*}} = \underline{V}_{ik}\ \tilde{S}_k\ \underline{V}_{2k}^{*}$$

$$= \sum_{j=1}^{J} S_{kj}\ e^{-j\omega_k \tau_j} \qquad \text{(H-4b)}$$

where

$$\tau_j = D_{2j} - D_{1j} \qquad \text{(H-4c)}$$

is the actual time delay difference between sensor array A_1 and A_2 of target j and

$$\tilde{S}_k = \overline{\underline{\beta}_k\ \underline{\beta}_k^{*}}$$

$$= \mathrm{diag}\{S_{k1}, S_{k2}, \ldots, S_{kJ}\} \qquad \text{(H-4d)}$$

is a diagonal signal spectral matrix. Therefore, the mean estimate must satisfy the equation

$$\overline{f(\tau)} = \sum_{k=-B}^{B} j\omega_k |h_k|^2\ \underline{V}_{1k}\ \tilde{S}_k\ \underline{V}_{2k}^{*}\ e^{j\omega_k \tau} = 0 \qquad \text{(H-5a)}$$

or using Equation (H-4b), we have

$$\overline{f(\tau)} = \sum_{j=1}^{J} \sum_{k=-B}^{B} j\omega_k |h_k|^2\ S_{kj}\ e^{j\omega_k(\tau-\tau_j)} = 0. \qquad \text{(H-5b)}$$

Equation (H-5b) can be manipulated to yield

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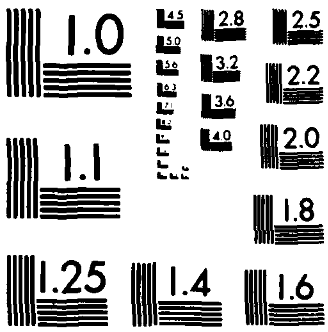

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$$\overline{f(\tau)} = \frac{\partial}{\partial\tau} \sum_{j=1}^{J} R_j(\tau - \tau_j) = 0 \qquad \text{(H-5c)}$$

where

$$R_j(\tau) \triangleq \sum_{k=-B}^{B} |h_k|^2 \, S_{kj} \, e^{j\omega_k \tau} \qquad \text{(H-5d)}$$

is the auto-correlation of target j.

Denoting the total power for target j by

$$P_j = \sum_{k=-B}^{B} S_{kj}, \qquad \text{(H-6a)}$$

the total received power by the sensor array as

$$P = \sum_{j=1}^{J} P_j + N; \; N = \sum_{k=-B}^{B} N_k, \qquad \text{(H-6b)}$$

and the normalized total correlation by

$$\rho(\tau) \triangleq \frac{\overline{f(\tau)}}{P}, \qquad \text{(H-6c)}$$

then Equation (H-5c) can be rewritten as

$$\rho(\tau) = \sum_{j=1}^{J} a_j \frac{\partial}{\partial\tau} \rho_j(\tau - \tau_j) = 0 \qquad \text{(H-7a)}$$

where

$$a_j = P_j/P \tag{H-7b}$$

and

$$\rho_j(\tau) = R_j(\tau)/P_j. \tag{H-7c}$$

In general, Equation (H-7a) must be solved numerically; however, for the case in which the interference is close to the true target of interest, then one can write

$$\rho_j(\tau - \tau_j) \simeq \rho_j(0) + \rho_j'(0)(\tau - \tau_j) + \frac{\rho_j''(0)}{2}(\tau - \tau_j)^2 \tag{H-8a}$$

where $\rho_j'(0)$ and $\rho_j''(0)$ denote the first and second derivative.

But

$$\rho_j'(0) = \left(\sum_{k=-B}^{B} j\omega_k |h_k|^2 S_{kj}\right)/P_j = 0 \tag{H-8b}$$

and

$$\rho_j''(0) = -\left(\sum_{k=-B}^{B} \omega_k^2 |h_k|^2 S_{kj}\right)/P_j$$

$$= -\left(\frac{T}{\pi}\int_0^B \omega_k^2 |h\omega|^2 S_j(\omega)\, d\omega\right)/P_j$$

$$= -W_j/P_j \tag{H-8c}$$

$$= - \frac{T(S_j/N)(S_1/N)}{\pi P_j(1 + 2\ S_1/N)} \int_{\omega_1}^{\omega_2} \omega^2\ d\omega \qquad \text{(H-8d)}$$

where in the last equation the signals and noise are assumed flat and bandlimited.

Therefore, using Equation (H-8a) in Equation (H-7a) yields

$$\sum_{j=1}^{J} a_j\ \rho_j''(0)(\tau - \tau_j) = 0\ .$$

Thus, the mean estimate of τ_1 is

$$\hat{\tau}_1 = \left(\sum_{j=1}^{J} a_j\ \rho_j''(0)\tau_j \right) \left(\sum_{j=1}^{J} a_j\ \rho_j''(0) \right)^{-1}$$

$$= \left(\sum_{j=1}^{J} W_j\ \tau j \right) \left(\sum_{j=1}^{J} W_j \right)^{-1} \qquad \text{(H-8e)}$$

where

$$W_j = \frac{T}{\pi} \int_{0}^{B} \omega^2 |h(\omega)|^2\ S_j(\omega)\ d\omega\ . \qquad \text{(H-8f)}$$

Now using Equations (H-8e) and (H-7b) in Equation (H-8d) yields

$$\hat{\tau}_1 = \left[\sum_{j=1}^{J} (S_j/N)\tau_i \right] \left[\sum_{j=1}^{J} (S_j/N) \right]^{-1} \qquad \text{(H-9)}$$

Thus, the time delay estimate from a second expansion of the correlation function is a weighted linear combination of time delays from every target. For the two-target case, let the signal of interest be $S_1 = S$, interference $S_2 = I$, then Equation (H-9) yields

$$\hat{\tau}_1 = \frac{S}{S+I}\,\tau_1 + \frac{I}{S+I}\,\tau_2 \ . \tag{H-10}$$

The bias in general is

$$\begin{aligned} b_1 &= \hat{\tau}_1 - \tau_1 \\ &= \left[\sum_{j=1}^{J} (S_j/N)(\tau_i - \tau_1)\right] \left[\sum_{j=1}^{J} (S_j/N)\right]^{-1} \end{aligned} \tag{H-11}$$

and for the two-target case

$$b_1 = \frac{1}{1 + S/I}\,(\tau_2 - \tau_1). \tag{H -12}$$

We next derive the expression for the variance. Using the linearization procedure, it is easy to verify that the resulting variance of the time delay estimate from seeking the null of the function $f(\tau)$ is given by

$$\mathrm{VAR}(\hat{\tau}_1) = \frac{\overline{f(\hat{\tau}_1)^2}}{\left(\overline{\dfrac{\partial f(\hat{\tau}_1)}{\partial \tau}}\right)^2} \tag{H-13a}$$

where $\hat{\tau}_1$ is chosen such that

$$\overline{f(\hat{\tau}_1)} = \sum_{k=-B}^{B} j\omega_k |h_k|^2 \, \overline{\alpha_{1k} \, \alpha_{2k}^*} \, e^{j\omega_k \hat{\tau}_1} = 0. \tag{H-13b}$$

But from Equation (H-3a) and (H-5a-d), we have

$$\overline{f(\hat{\tau}_1)^2} = \sum_{k=-B}^{B} \sum_{\ell=-B}^{B} (j\omega_k)(j\omega\ell) |h_k|^2 |h_\ell|^2 \, \overline{\alpha_{1k} \, \alpha_{2k}^* \, \alpha_{1\ell} \, \alpha_{2\ell}^*} \, e^{j(\omega_k + \omega_\ell)\hat{\tau}_1} \tag{H-14a}$$

and

$$\overline{\frac{\partial f(\hat{\tau}_1)}{\partial \tau}} = \sum_{k=-B}^{B} (j\omega_k)^2 |h_k|^2 \, \overline{\alpha_{1k} \, \alpha_{2k}^*} \, e^{j\omega_k \hat{\tau}_1}. \tag{H-14b}$$

Now using the following relations

$$\overline{\alpha_{1k} \, \alpha_{2k}^*} = \underline{V}_{1k} \, \tilde{S}_k \, \underline{V}_{2k} = \sum_{j=1}^{J} S_{kj} \, e^{-j\omega_k \tau_j} \tag{H-15a}$$

$$\overline{\alpha_{1k} \, \alpha_{2k}^* \, \alpha_{1\ell} \, \alpha_{2\ell}^*} = \overline{\alpha_{1k} \, \alpha_{2k}^*} \; \overline{\alpha_{1\ell} \, \alpha_{2\ell}^*} + \overline{\alpha_{1k} \, \alpha_{1\ell}} \; \overline{\alpha_{2k}^* \, \alpha_{2\ell}^*} + \overline{\alpha_{1k} \, \alpha_{2\ell}^*} \; \overline{\alpha_{2k}^* \, \alpha_{1\ell}}$$

$$= \overline{\alpha_{1k} \, \alpha_{2k}^*} \; \overline{\alpha_{1\ell} \, \alpha_{2\ell}^*} + (\underline{V}_{1k} \, \tilde{S}_k \, \underline{V}_{1k}^* + N_k)^2 \delta_{k+\ell}$$

$$+ (\underline{V}_{1k} \, \tilde{S}_k \, \underline{V}_{2k}^*)^2 \, \delta_{k-\ell} \tag{H-15b}$$

Equations (H-14a) and (H-14b) can be simplified to

$$\overline{f(\tau_1)^2} = \sum_{k=-B}^{B} \omega_k^2 |h_k|^4 \left\{ (\underline{V}_{1k}\ \tilde{S}_k\ \underline{V}_{1k}^* + N_k)^2 - (\underline{V}_{1k}\ \tilde{S}_k\ \underline{V}_{2k}^*\ e^{j\omega_k \hat{\tau}_1})^2 \right\}$$

(H-16a)

$$\overline{\frac{\partial f(\tau_1)}{\partial \tau}} = - \sum_{k=-B}^{B} \omega_k^2 |h_k|^2\ \underline{V}_{1k}\ \tilde{S}_k\ \underline{V}_{2k}^*\ e^{j\omega_k \tau_1}. \qquad \text{(H-16b)}$$

Substituting Equations (H-16a) and (H-16b) in (H-13a) yields the desired expression for the variance of the $\hat{\tau}_1$ estimate:

$$\mathrm{VAR}(\hat{\tau}_1) = \frac{\sum_{k=-B}^{B} \omega_k^2 |h_k|^4 \left\{ (\underline{V}_{1k}\ \tilde{S}_k\ \underline{V}_{1k}^* + N_k)^2 - (\underline{V}_{1k}\ \tilde{S}_k\ \underline{V}_{2k}^*\ e^{j\omega \hat{\tau}_1})^2 \right\}}{\left[\sum_{k=-B}^{B} \omega_k^2 |h_k|^2 (\underline{V}_{1k}\ \tilde{S}_k\ \underline{V}_{2k}^*)\ e^{j\omega_k \tau_1} \right]^2}$$

(H-17a)

$$= \frac{2\pi}{T} \frac{\int_B \omega^2 |h|^4 \left[(\underline{V}_1\ \tilde{S}\ \underline{V}_1^* + N)^2 - \left(\underline{V}_1\ \tilde{S}\ \underline{V}_2^*\ e^{j\omega \hat{\tau}_1} \right)^2 \right] d\omega}{\left(\int_B \omega^2 |h|^2\ \underline{V}_1\ \tilde{S}\ \underline{V}_2^*\ e^{j\omega \hat{\tau}_1} d\omega \right)^2}. \qquad \text{(H-17b)}$$

Recall that

$$\underline{V}_1\ \tilde{S}\ \underline{V}_1^* + N = N + \sum_j S_j \qquad \text{(H-18a)}$$

$$\underline{V}_1\ \tilde{S}\ \underline{V}_2^*\ e^{j\omega \hat{\tau}_1} = \sum_j S_j\ e^{j\omega \Delta_j} \qquad \text{(H-18b)}$$

where $\Delta_j \triangleq \hat{\tau}_1 - \tau_j$, then Equation (H-17b) can be rewritten as

$$\mathrm{VAR}(\hat{\tau}_1) = \frac{2\pi}{T} \frac{\int_B \left(\left(N + \sum_j S_j\right)^2 - \left(\sum_j S_j\, e^{j\omega\Delta_j}\right)^2\right) \omega^2 |h|^4 \, d\omega}{\left[\int_B \left(\sum_j S_j\, e^{j\omega\Delta_j}\right) \omega^2 |h|^2 \, d\omega\right]^2} \; . \qquad \text{(H-19)}$$

For the special case where

$$S_j(\omega) = \begin{cases} S_j; \; \omega_1 \le \omega \le \omega_2 \text{ and } -\omega_2 \le \omega \le -\omega_1 \\ 0 \; ; \text{ otherwise} \end{cases}$$

$$N(\omega) = \begin{cases} N; \; \omega_1 \le \omega \le \omega_2 \text{ and } -\omega_2 \le \omega \le -\omega_1 \\ 0; \text{ otherwise} \end{cases}$$

Equation (H-19) reduces to

$$\mathrm{VAR}(\hat{\tau}_1) = \frac{2\pi}{T} \frac{\left(N + \sum_j S_j\right)^2 R(0) - \sum_i \sum_j S_i\, S_j\, R(\Delta_i + \Delta_j)}{\left(\sum_j S_j\, R(\Delta_j)\right)^2}$$

$$= \frac{2\pi}{T} \left[\frac{1 - \sum_i \sum_j a_i a_j\, \rho(\Delta_i + \Delta_j)}{2\left(\sum_j a_j\, \rho(\Delta_j)\right)^2}\right] R^{-1}(0) \qquad \text{(H-20a)}$$

where

$$a_i = S_i \left(N + \sum_j S_j \right)^{-1} \tag{H-20b}$$

$$R(\tau) = \int_{\omega_1}^{\omega_2} \omega^2 \cos \omega\tau \, d\omega \tag{H-20c}$$

$$\rho(\tau) = R(\tau)/R(0) \tag{H-20d}$$

$$= \begin{cases} 1 \ ; \text{ if } \tau = 0 \\ \dfrac{1}{R(0)} \left[\dfrac{2\omega \cos \omega\tau}{\tau^2} + \dfrac{(\omega\tau)^2 - 2}{\tau^3} \sin \omega\tau \right]_{\omega_1}^{\omega_2} \ ; \text{ if } \tau \neq 0. \end{cases} \tag{H-20e}$$

For the two-target case, Equation (H-20) becomes

$$\mathrm{VAR}(\hat{\tau}_1) = \frac{\pi}{T} \left\{ \frac{1 - [a_1^2 \, \rho(2\Delta_1) + a_2^2 \, \rho(2\Delta_2) + 2a_1 a_2 \, \rho(\Delta_1 + \Delta_2)]}{2[a_1 \, \rho(\Delta_1) + a_2 \, \rho(\Delta_2)]^2} \right\} R^{-1}(0) \ . \tag{H-21a}$$

Note that

$$a_1 = \frac{S/N}{1 + S/N + I/N} \tag{H-21b}$$

$$a_2 = \frac{I/N}{1 + S/N + I/N} \tag{H-21c}$$

where we have let $S_1 = S$, $S_2 = I$ be the signal and the interference power spectrum, respectively. If the interference is far away from the target in time delay, Equation (H-21) becomes

$$\mathrm{VAR}_{\infty}(\hat{\tau}_1) = \lim_{\substack{\Delta_1 \to 0 \\ \Delta_2 \to \infty}} \mathrm{VAR}(\hat{\tau}_1)$$

$$= \frac{2\pi}{T}\left(\frac{1 - a_1^2}{2a_1^2}\right) R^{-1}(0) \; . \tag{H-22}$$

Therefore, the normalized variance is

$$\frac{\mathrm{VAR}(\hat{\tau}_1)}{\mathrm{VAR}_{\infty}(\hat{\tau}_1)} = \left(\frac{a_1^2}{1 - a_1^2}\right)\left[\frac{1 - (a_1^2\, \rho(2\Delta_1) + a_2^2\, \rho(2\Delta_2) + 2a_1 a_2\, \rho(\Delta_1 + \Delta_2)}{2(a_1\, \rho(\Delta_1) + a_2\, \rho(\Delta_2))}\right] . \tag{H-23}$$

Appendix I

CALCULATION OF MULTITARGET GENERALIZED CROSS-CORRELATION COVARIANCE

The calculation of the covariance matrix resulting from discrete observation of the Generalized Cross-Correlation (GCC) function in the presence of multitarget interference is presented. The results presented here are obtained from a frequency domain formulation.

Let the noisy observed waveforms from two sensors in the presence of J targets be represented by

$$y_1(t) = \sum_{i=1}^{J} s_i(t) + n_1(t) \tag{I-1}$$

$$y_2(t) = \sum_{i=1}^{J} s_i(t + D_i) + n_2(t) \tag{I-2}$$

where $s_i(t)$ is the ith target signal waveform and $n_1(t)$, $n_2(t)$ are the noise processes. It is assumed that signals and noises are zero mean, mutually uncorrelated, band-limited Gaussian processes.

Let the waveforms be sampled at a sampling rate of Δt seconds such that $T = N\ \Delta t$ seconds of the waveforms are observed. Discrete Fourier transforms of Equations (I-1) and (I-2) yield the equivalent frequency domain representation as

$$\alpha_{1k} = \sum_{i=1}^{J} \beta_{ki} + \eta_{1k} \tag{I-3}$$

$$\alpha_{2k} = \sum_{i=1}^{J} \beta_{ki}\, e^{j\omega_k D_i} + \eta_{2k} \tag{I-4}$$

where

$$\omega_k = 2\pi k/T$$

$$\alpha_{\ell k} = \frac{1}{N}\sum_{n=1}^{N} y_\ell(n\Delta t)\, e^{-j(2\pi nk/N)} \;; \qquad \ell = 1,\ 2 \tag{I-5}$$

$$\beta_{ki} = \frac{1}{N}\sum_{n=1}^{N} s_i(n\Delta t)\, e^{-j(2\pi nk/N)} \;; \qquad i = 1,\ 2,\ \ldots\ J \tag{I-6}$$

and

$$\eta_{\ell k} = \frac{1}{N}\sum_{n=1}^{N} n_\ell(n\Delta t)\, e^{-j(2\pi nk/N)} \;; \qquad \ell = 1,\ 2\ . \tag{I-7}$$

The GCC is obtained from the following:

$$R(\tau) = \sum_{k=-B}^{B} \alpha_{1k}\, \alpha_{2k}^{*}\, |H_k|^2\, e^{j\omega_k \tau} \tag{I-8}$$

where $|H_k|^2$ is the spectral shaping filter.

Equation (I-8) is usually implemented via inverse FFT. Consequently, discrete observations of the GCC $R(\tau)$, $R(n\Delta t)$, are obtained. For simplicity, let $R(n) = R(n\Delta t)$, then we can write

$$R(n) = \sum_{k=-B}^{B} \alpha_{1k} \alpha^*_{2k} |H_k|^2 e^{j(2\pi nk/N)} . \tag{I-9}$$

We are interested in the auto covariance of R(n) and R(m), Λ_{nm}, for any n and m. In general, one can write

$$\Lambda_{nm} = \overline{R(n)\ R(m)} - \overline{R(n)}\ \overline{R(m)} . \tag{I-10}$$

Now substituting Equation (I-9) into (I-10) and simplifying yields

$$\Lambda_{nm} = \sum_{k=-B}^{B} \sum_{\ell=-B}^{B} \left\{ \left(\overline{\alpha_{1k}\ \alpha^*_{2k}\ \alpha_{1\ell}\ \alpha^*_{2\ell}} - \overline{\alpha_{1k}\ \alpha^*_{2k}}\ \overline{\alpha_{1\ell}\ \alpha^*_{2\ell}} \right) \right.$$

$$\left. |H_k|^2\ |H_\ell|^2\ e^{j(2\pi/N)(kn+\ell m)} \right\} \tag{I-11}$$

Since α_{ik} for all i and k are zero mean complex Gaussian random variables uncorrelated for different frequency ω_k, one can write (using the fourth order product moment formula)

$$\overline{\alpha_{1k}\ \alpha^*_{2k}\ \alpha_{1\ell}\ \alpha^*_{2\ell}} = \overline{\alpha_{1k}\ \alpha^*_{2k}}\ \overline{\alpha_{1\ell}\ \alpha^*_{2\ell}} + \overline{\alpha_{1k}\ \alpha_{1\ell}}\ \overline{\alpha^*_{2k}\ \alpha^*_{2\ell}} +$$

$$\overline{\alpha_{1k}\ \alpha^*_{2\ell}}\ \overline{\alpha^*_{2k}\ \alpha_{1\ell}} . \tag{I-12}$$

Substituting Equation (I-12) into (I-11) yields

$$\Lambda_{nm} = \sum_{k=-B}^{B} \sum_{\ell=-B}^{B} \left\{ \left(\overline{\alpha_{1k}\ \alpha_{1\ell}}\ \overline{\alpha^*_{2k}\ \alpha^*_{2\ell}} + \overline{\alpha_{1k}\ \alpha^*_{2\ell}}\ \overline{\alpha^*_{2k}\ \alpha_{1\ell}} \right) \right.$$

$$\left. |H_k|^2\ |H_\ell|^2\ e^{j(2\pi/N)(kn+\ell m)} \right\} . \tag{I-13}$$

Now using Equations (I-3) and (I-4), the quantities inside the parentheses in (I-13) can be evaulated as follows:

$$\overline{\alpha_{1k}\,\alpha_{1\ell}}\;\overline{\alpha^*_{2k}\,\alpha^*_{2\ell}} = \left(\sum_{i=1}^{J} S_{ki} + N_{1k}\right)\left(\sum_{i=1}^{J} S_{ki} + N_{2k}\right)\delta_{k+\ell} \tag{I-14}$$

$$\overline{\alpha_{1k}\,\alpha^*_{2\ell}}\;\overline{\alpha^*_{2k}\,\alpha_{1\ell}} = \left(\sum_{i=1}^{J} S_{ki}\, e^{-j\omega_k D_i}\right)^2 \delta_{k-\ell} \tag{I-15}$$

where $S_{ki} = \overline{\beta^2_{ki}}$, $N_{1k} = \overline{\eta^2_{1k}}$, and $N_{2k} = \overline{\eta^2_{2k}}$.

Finally, substituting Equations (I-14) and (I-15) into (I-13) yields

$$\Lambda_{nm} = \sum_{k=-B}^{B}\left\{\left(\sum_{i=1}^{J} S_{ki} + N_{1k}\right)\left(\sum_{i=1}^{J} S_{ki} + N_{2k}\right)\right\}|H_k|^4\, e^{j(2\pi k/N)(n-m)}$$

$$+ \sum_{k=-B}^{B}\left(\sum_{i=1}^{J} S_{ki}\, e^{-j\omega_k D_i}\right)^2 |H_k|^4\, e^{j(2\pi k/N)(n+m)} \;. \tag{I-16}$$

For a single target case with identical noise power spectral density, Equation (I-16) yields

$$\Lambda^{(1)}_{nm} = \sum_{k=-B}^{B}\left\{(S_k + N_k)^2\, e^{j(2\pi k/N)(n-m)}\right.$$

$$\left. + S_k^2\, e^{-j2\omega_k D}\, e^{j(2\pi k/N)(n-m)}\right\}|H_k|^4$$

$$= \sum_{k=-B}^{B} (S_k^2 + 2S_kN_k + N_k^2) \; |H_k|^4 \; e^{j(2\pi k/N)(n-m)}$$

$$+ \sum_{k=-B}^{B} |H_k|^4 \; S_k^2 \; e^{j(2\pi k/N)(n+m-2D/\Delta t)} \tag{I-17}$$

$$\simeq \frac{T}{2\pi} \left\{ \int_{-B}^{B} (S^2 + 2SN + N^2) \; |H|^4 \; e^{j\omega(n-m)\Delta t} \; d\omega \right.$$

$$\left. + \int_{-B}^{B} |H|^4 \; S^2 \; e^{j\omega((n+m)\Delta t - 2D)} \; d\omega \right\} \tag{I-18}$$

where S, N, and H denote the continuous power spectra.

For the case of uniform spectral shaping, i.e., $|H|^2 = 1$, we obtain

$$\Lambda_{nm}^{(1)} = \rho_{SS}(n - m) + 2\rho_{SN}(n - m) + \rho_{NN}(n - m) + \rho_{SS}[(n + m) - 2D/\Delta t] \tag{I-19}$$

where if $\Phi_{XX}(\omega)$ is the power spectrum of X, then $\rho_{XX}(\tau)$ is given by

$$\rho_{XX}(\tau) = \frac{T}{2\pi} \int_{-B}^{B} \Phi_{XX}^2(\omega) \; e^{j\omega\tau} \; d\omega \; . \tag{I-20}$$

For a two-target case with equal noise power spectra, Equation (I-16) reduces to:

$$\Lambda_{nm}^{(2)} = \sum_{k=-B}^{B} (S_{k1} + S_{k2} + N_k)^2 \; |H_k|^4 \; e^{j(2\pi k/N)(n-m)}$$

$$+ \sum_{k=-B}^{B} \left(S_{k1} \; e^{-j\omega_k D_1} + S_{k2} \; e^{-j\omega_k D_2}\right)^2 \; |H_k|^4 \; e^{j(2\pi k/N)(n+m)} \qquad \text{(I-21)}$$

$$= \sum_{k=-B}^{B} |H_k|^4 \left\{ \left[S_{k1}^2 + S_{k2}^2 + N_k^2 + 2(S_{k1}S_{k2} + S_{k1}N_k + S_{k2}N_k)\right] \right.$$

$$e^{j(2\pi k/N)(n-m)} + \left[S_{k1}^2 \; e^{-j2\omega_k D_1} + 2S_{k1}S_{k2} \; e^{-j\omega_k (D_1+D_2)} \right.$$

$$\left. \left. + \; S_{k2}^2 \; e^{-j2\omega_k D_2}\right] e^{j(2\pi k/N)(n+m)} \right\}. \qquad \text{(I-22)}$$

Hence the continuous approximation is given by

$$\Lambda_{nm}^{(2)} \approx \frac{T}{2\pi} \int_{-B}^{B} |H|^4 \left\{ \left[S_1^2 + S_2^2 + N^2 + 2(S_1S_2 + S_1N + S_2N)\right] \; e^{j\omega(n-m)\Delta t} \right.$$

$$+ \left[S_1^2 \; e^{-j2\omega D_1} + 2S_1S_2 \; e^{-j\omega(D_1+D_2)} + S_2^2 \; e^{-j2\omega D_2}\right]$$

$$\left. e^{j\omega(n+m)\Delta t} \right\} d\omega \; . \qquad \text{(I-23)}$$

Now for uniform spectral shading $|H|^2 = 1$ and using the definition of Equation (I-20) in (I-23) yields

$$\Lambda_{nm}^{(2)} = \rho_{S_1S_1}(n - m) + \rho_{S_2S_2}(n - m) + \rho_{NN}(n - m)$$

$$+ 2\left[\rho_{S_1S_2}(n - m) + \rho_{S_1N}(n - m) + \rho_{S_2N}(n - m)\right]$$

$$+ \rho_{S_1S_2}\ [(n + M) - 2D_1/\Delta t] + 2\rho_{S_1S_2}\ [(n + M) - (D_1 + D_2)/\Delta t]$$

$$+ \rho_{S_2S_2}\ [(n + M) - 2D_2/\Delta t]\ . \qquad \text{(I-24)}$$

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